International Journal of Theoretical Physics, Vol. 40, No. 12, December 2001 (© 2001)

A Crucial Ingredient of Inflation

Valerio Faraoni[1–3]

Received March 10, 2001

Nonminimal coupling of the inflaton field to the Ricci curvature of spacetime is generally unavoidable, and the paradigm of inflation should be generalized by including the corresponding term $\xi R\phi^2/2$ in the Lagrangian of the inflationary theory. This paper reports on the status of the programme of generalizing inflation. First, the problem of finding the correct value (or set of values) of the coupling constant ξ is analyzed; the result has important consequences for the success or failure of inflationary scenarios. Then, the slow-roll approximation to generalized inflation is studied. Both the unperturbed inflating universe models and scalar/tensor perturbations are discussed, and open problems are pointed out.

1. INTRODUCTION

Cosmic inflation is a period of accelerated expansion of the universe during its early phase: provided that inflation proceeds for a sufficiently long time (such that the cosmic expansion in the inflationary period is about 60 e-folds) and that physical criteria for a successful description of the universe (Kolb and Turner, 1994; Linde, 1990) are met, inflation solves the classic problems of the standard big bang cosmology (the horizon, flatness, and monopole problem, Kolb and Turner, 1994; Linde, 1990). In addition, inflation provides, as a bonus, a mechanism (quantum fluctuations of the inflaton field) to generate density perturbations, the seeds of structures observed in the universe today (galaxies, clusters, and superclusters).

Nowadays, this last aspect is regarded as the main motivation to pursue research on inflation (e.g., Liddle, 1999). There are many scenarios of inflation, but no "standard model" is universally accepted: inflation has been called a "paradigm in search of a model." In the vast majority of inflationary scenarios, the cosmic acceleration is driven by one (or more) scalar field(s): although there are exceptions

[1] Research Group in General Relativity (RggR), Université Libre de Bruxelles, Bruxelles, Belgium.
[2] INFN-Laboratori Nazionali di Frascati, Frascati, Roma, Italy.
[3] To whom correspondence should be addressed at Department of Physics, University of Northern British Columbia, 3333 University Way, Prince George.

(e.g., the scenario of Starobinsky, 1980a,b), a scalar field is sometimes added even to these scenarios in order to "help" inflation (Maeda, 1989).

The inflaton field ϕ satisfies the Klein–Gordon equation: for reasons explained ahead, when generalizing the latter from Minkowski space to a curved space, one needs to introduce, in general, a nonminimal coupling term between the scalar ϕ and the Ricci curvature of spacetime R as follows:

$$\Box\phi - \frac{dV}{d\phi} - \xi R\phi = 0, \tag{1.1}$$

where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is d'Alembert's operator on a curved space, $V(\phi)$ is the scalar field self-interaction potential, and ξ is a dimensionless coupling constant. The classic works on inflation neglected the $-\xi R\phi$ term in Eq. (1.1) (which is equivalent to assume that $\xi = 0$); hereafter, this theory is called *ordinary inflation*, as opposed to *generalized inflation*, which corresponds to $\xi \neq 0$. As explained in the next section, almost always the introduction of a nonminimal (i.e., $\xi \neq 0$ in Eq. (1.1)) coupling is not an option; rather, it is unavoidable. This fact is not well known to cosmologists and has profound consequences for the physics of the inflaton. Then, given the unavoidability of nonminimal coupling (hereafter NMC), one needs to rethink inflation by appropriately including terms corresponding to $\xi \neq 0$ in the relevant equations. This was already done for *specific* inflationary scenarios by a number of authors (Barroso *et al.*, 1992; Bassett and Liberati, 1998; Fakir *et al.*, 1992; Fakir and Habib, 1993; Fakir and Unruh, 1990a,b; Futamase *et al.*, 1989; Futamase and Tanaka, 1999; Garcia-Bellido and Linde, 1995; Hwang and Noh, 1998; Komatsu and Futamase, 1998, 1999; Laycock and Liddle, 1994; Lee *et al.*, 1999; Madsen, 1988; Makino and Sasaki, 1991; Salopek *et al.*, 1989; Starobinsky, 1981); however, the approach adopted was largely one in which the coupling constant ξ is regarded as an extra parameter of inflation that can be used at one's will in order to cure preexisting problems of the inflationary scenario. To make an example, chaotic inflation with quartic self-interaction $V(\phi) = \lambda\phi^4$ and $\xi = 0$ is fine-tuned: the amplitude of anisotropies of the cosmic microwave background requires $\lambda \leq 10^{-12}$, a figure that makes the scenario uninteresting from the point of view of particle physics which originally motivated it. The fine-tuning is significantly reduced if one introduces nonminimal coupling with $\xi < 0$ and $|\xi| \simeq 10^4$ (Fakir and Unruh, 1990a; Makino and Sasaki, 1991); the price to pay for reducing the fine-tuning of λ is the fine-tuning of ξ. We disagree with the philosophy of this approach because the coupling constant ξ has, in general, a well-defined value in nature[4] and is not an extra free parameter of the theory. In Section 3 we review the known prescriptions for the value of the coupling constant ξ and make clear that, not only $\xi \neq 0$ in the general case, but also that fine-tuning ξ is not

[4] Exceptions, discussed later, are the cases in which ξ is a running coupling in GUT theories, or when first loop corrections are taken into account.

a possibility. We then proceed to analyze the consequences of including NMC into the equations of inflation. The study necessarily proceeds at two levels: first, one has to consider the *unperturbed* background universe; and then one continues with the study of scalar and tensor *perturbations* of the fixed inflationary background universe. The amplitudes and spectra of perturbations are very important since they leave a detectable imprint in the cosmic microwave background.

Temperature anisotropies in the sky, likely the fingerprints of inflation, have been discovered by the COBE satellite (Smoot *et al.*, 1992) and their experimental study is one of the primary goals of current cosmology. Major improvements will come with the MAP[5] and PLANCK[6] satellites to be launched, respectively, in the years 2001 and 2007.

In this paper we approach the task of reformulating generalized inflation (i.e., including the $\xi \neq 0$ terms in the picture) from a *general* point of view, that is, we do not adopt a specific inflationary scenario. The results for the unperturbed universe are presented in Sections 3 and 4.

A special role is played by the slow-roll approximation: apart from two exceptions (power-law inflation and the string-inspired, toy model of Easther (1996)—see also Martin and Schwarz (2001)), one cannot exactly solve the equations of inflation (both unperturbed and perturbed), and one needs to resort to the slow-roll approximation. The latter has been discussed in great detail for minimal (i.e., $\xi = 0$) coupling (see Lidsey *et al.*, 1997 for a recent review), and is much needed also in the case of nonminimal, coupling, for which the equations are even more complicated. Slow-roll generalized inflation is discussed in Section 4. The study of scalar and tensor perturbations with nonminimal coupling is the subject of Section 5, where previous results are reviewed and completed to obtain explicit formulas for the observables of inflation. de Sitter solutions play the role of attractors of inflationary solutions in the phase space for generalized, as well as ordinary, inflation; it is this fact that ultimately gives meaning to the slow-roll approximation. Section 6 contains a list of open problems and the conclusions.

2. NONMINIMAL COUPLING OF THE SCALAR FIELD

The action of gravity plus a nonminimally coupled scalar field as matter is

$$S = \int d^4x\, \sqrt{-g} \left[\frac{R}{2\kappa} - \frac{1}{2} \nabla^c \phi \nabla_c \phi - V(\phi) - \frac{\xi}{2} R\phi^2 \right], \tag{2.1}$$

where $\kappa \equiv 8\pi G$, G is Newton's constant and, apart from minor differences, we adopt the notations and conventions[7] of Wald (1984).

[5] MAP homepage: http://www.map.gsfc.nasa.gov/

[6] PLANCK homepage: http://astro.estec.esa.nl/SA-general/Projects/Planck

[7] The metric signature is $-+++$ and G denotes Newton's constant. The speed of light and Planck's constant assume the value unity and $m_{\rm pl} = G^{-1/2}$ is the Planck mass. The components of the Ricci

2.1. Why $\xi \neq 0$?

The nonminimal coupling of the scalar ϕ described by Eq. (1.1) was apparently introduced for the first time by Chernikov and Tagirov (1968), although it is better known from the work of Callan *et al.* (1970). Why should one consider $\xi \neq 0$? The answer is manyfold: a nonzero ξ is generated by quantum corrections even if it is absent in the classical action (Birrell and Davies, 1980a,b; Ford, 1987; Ford and Toms, 1982; Nelson and Panangaden, 1982; Parker and Toms, 1985a). If one prepares a classical theory with $\xi = 0$, renormalization shifts it to one with $\xi \neq 0$. Even though the shift is small, it can have a tremendous effect on an inflationary scenario. The classical example of this effect is related to chaotic inflation (Futamase and Maeda, 1989): the shift $\xi = 0 \rightarrow \xi_{\text{renormalized}} \approx 10^{-1}$ (a typical value predicted by renormalization, Allen, 1993; Ishikawa, 1983) is sufficient to ruin the chaotic inflationary scenario with potential $V = \lambda\phi^4$ (Futamase *et al.*, 1989; Futamase and Maeda, 1989).

Another reason to include a $\xi \neq 0$ term in the coupled Einstein–Klein–Gordon equations is that it is required by renormalization of the theory (this was the motivation for the introduction of NMC by Callan *et al.* (1970). It has also been argued (see ahead) that a NMC term is expected at high curvatures (Ford, 1987; Ford and Toms, 1982), and that classicalization of the universe in quantum cosmology requires $\xi \neq 0$ (Okamura, 1998). A pragmatic point of view would be that, since NMC may be crucial for the success or failure of inflation (Abbott, 1981; Faraoni, 1996, 1998; Futamase *et al.*, 1989; Futamase and Maeda, 1989), one better take it into account and decide *a posteriori* whether its presence is negligible or not.

In relativity, it turns out that any value of ξ different from $1/6$ ("conformal coupling," the value that makes the Klein–Gordon equation (1.1), and the physics of ϕ, conformally invariant if $V = 0$ or $V = \lambda\phi^4$, Wald, 1984) spoils the Einstein equivalence principle and is therefore not admissible (Faraoni and Sonego, 1994; Grib and Poberii, 1995; Grib and Rodrigues, 1995; Sonego and Faraoni, 1993).

Whichever point of view one adopts, with motivations arising in a range of areas as wide as quantum field theory in curved spaces, wormholes (Coule, 1992; Coule and Maeda, 1990; Halliwell and Laflamme, 1989), black holes (Hiscock, 1990; van der Bij and Radu, 2000), boson stars (Jetzer, 1992; van der Bij and Gleiser, 1987), specific inflationary scenarios, a pure relativist's approach, or merely a pragmatic one, the message is that in general it is wise not to ignore the NMC term by setting $\xi = 0$, as done in ordinary inflation. Although the inclusion of NMC makes the analysis considerably more difficult, and it was unknown in the early, pioneering days of inflationary theory, the times are mature for the inclusion of NMC in the theory.

tensor are given in terms of the Christoffel symbols $\Gamma^{\delta}_{\alpha\beta}$ by $R_{\mu\rho} = \Gamma^{\nu}_{\mu\rho,\nu} - \Gamma^{\nu}_{\nu\rho,\mu} + \Gamma^{\alpha}_{\mu\rho}\Gamma^{\nu}_{\alpha\nu} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\nu}_{\alpha\mu}$, and $\Box \equiv g^{ab}\nabla_a\nabla_b$.

2.2. What is the Value of ξ?

It is plausible that the value of the coupling constant ξ be fixed by the physics of the problem, and not be left to the choice of the theoretician as a free parameter. A particle physicist's answer to the question "what is the value of ξ?" differs according to the theory of the scalar field employed.

If ϕ is a Goldstone boson in a theory with a spontaneously broken global symmetry, then $\xi = 0$ (Voloshin and Dolgov, 1982). If the scalar field ϕ is associated to a composite particle, the value of ξ is fixed by the dynamics of its components. In particular, in the large N approximation to the Nambu–Jona–Lasinio model, the value $\xi = 1/6$ was found (Hill and Salopek, 1992). In the O(N)-symmetric model with quartic self interaction, in which the constituents of the ϕ-particle are themselves bosons, ξ depends on the coupling constants ξ_i of the elementary scalars (Reuter, 1994). In Einstein's gravity with the potential

$$V(\phi) = V_0 + \frac{m^2}{2}\phi^2 + \frac{\eta}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4 \tag{2.2}$$

and back-reaction, the value $\xi = 0$ was found (Hosotani, 1985; Parker and Toms, 1985b). Higgs fields in the standard model have values of ξ in the range $\xi \leq 0$, $\xi \geq 1/6$ (Hosotani, 1985).

A great deal of results is available in the renormalization group approach to quantum field theory in curved spaces. It is shown in Buchbinder (1986), Buchbinder *et al.* (1986), Buchbinder and Odintsov (1983, 1985), Elizalde and Odintsov (1994), Muta and Odintsov (1991), and Odintsov (1991) that in asymptotically free GUTs, depending on the gauge group employed (SU(2), SU(N), SO(N), ...) and on the matter content, ξ is a running coupling that converges to $1/6$ (asymptotic conformal invariance), or to a value ξ_0 determined by the initial conditions (usually this occurs for supersymmetric GUTs) or (formally), $|\xi(\tau)| \rightarrow +\infty$. The last behaviour is often characteristic of large gauge groups (SU(10), SO(10), ...). Here τ is a renormalization group parameter, with $\tau \rightarrow +\infty$ corresponding to strong curvature and early universe situations. In Buchbinder *et al.* (1992) it was shown that also in asymptotically free GUTs with SU(5) as the gauge group, $|\xi(\tau)| \rightarrow +\infty$. Finite GUTs (another class of GUTs) behave similarly to asymptotically free GUTs, with $\xi(\tau)$ converging to $1/6$, or to an initial value ξ_0 (e.g., for $N = 4$ supersymmetry), or to infinity. Moreover, for finite GUTs the convergence of $\xi(\tau)$ to its asymptotic value as $\tau \rightarrow +\infty$ is much faster than in asymptotically free GUT models (indeed, the convergence is exponentially fast, Buchbinder, 1989, 1992). Hence, the asymptotic value of ξ in the early universe strongly depends on the choice of the specific GUT and of its gauge group and matter content.

The problem of the value of ξ in this context is not an easy one, as is clear from the example case of the simple $\lambda\phi^4$ theory. The latter is asymptotically free in the infrared limit ($\tau \to -\infty$), which does not correspond to high curvature. Nevertheless, it was shown in Buchbinder (1986), Buchbinder *et al.* (1986), Buchbinder and Odintsov (1983, 1985), Elizalde and Odintsov (1994), Muta and Odintsov (1991), and Odintsov (1991) that $\xi(\tau) \to 1/6$ as $\tau \to -\infty$. In the limit $\tau \to +\infty$ of strong curvatures, one cannot answer the question of the asymptotic value of $\xi(\tau)$ since the theory is contradictory (not asymptotically free) in this limit. Nevertheless, an *exact* renormalization group approach to the $\lambda\phi^4$ theory shows that $\xi = 1/6$ is indeed a stable fixed point of the exact renormalization group (Parker and Toms, 1985b).

So far, controversies on these results only arose for a restricted class of specific models (Bonanno, 1995; Bonanno and Zappalá, 1997). The divergence of the running coupling ξ as the energy scale and the curvature and temperature increase going back in time in the history of the universe, has been introduced in cosmology (Hill and Salopek, 1992) and exploited to make the chaotic inflationary scenario with $\xi < 0$ more plausible in the cases in which $|\xi(\tau)| \to +\infty$ (Futamase and Tanaka, 1999). The divergence of the coupling ξ is also crucial for the success of the so-called "geometric reheating" of the universe after inflation (Bassett and Liberati, 1998), in which particles are created due to the strong coupling of the inflaton to the Ricci curvature R, instead of the usual coupling of ϕ to other fields.

First loop corrections to the classical theory make ξ likely to be a running parameter which depends on the Ricci curvature: in Ford and Toms (1982) and Parker and Toms (1985b) the effective coupling

$$\xi_{\text{eff}} = \xi + \frac{\lambda}{16\pi^2}\left(\xi - \frac{1}{6}\right)\ln s, \tag{2.3}$$

was found for the self-interaction potential $\lambda\phi^4/4!$, where s is a parameter that tends to zero in the renormalization group approach. In practice, this amounts to have the effective coupling

$$\xi_{\text{eff}} \propto \ln(Rl^2), \tag{2.4}$$

where l^{-1} is a renormalization mass (Ford, 1987).

To the best of our knowledge, no prescriptions for the value of ξ other than those reviewed were proposed in the high energy physics literature. Instead, a strong prescription comes from relativity.

In general relativity (and in all other metric theories of gravity in which ϕ is a nongravitational field[8]), the only value of ξ allowed by the Einstein equivalence principle (Will, 1993) is the conformal coupling $1/6$. However, the derivation of this result (Faraoni and Sonego, 1994; Sonego and Faraoni, 1993) has nothing to

[8] For example, the Brans–Dicke scalar field is part of the gravitational sector of the theory together with the metric tensor g_{ab}, hence it is a *gravitational* scalar field.

do with conformal invariance, conformal transformations, or conformal flatness of the spacetime metric g_{ab}. It arises in the study of wave propagation and tails of scalar radiation (violations of the Huygens' principle) in curved spaces. This field of mathematical physics is rather far from cosmology and, *a priori*, it is unlikely to contribute to cosmology, but this is not the case. Before getting into details, let us anticipate the main idea: one imposes that the structure of tails of ϕ (which satisfies the wave equation (1.1)) becomes closer and closer to that occurring in Minkowski space as the curved manifold is progressively approximated by its tangent space. This is the Einstein equivalence principle (Will, 1993) applied to the physics of ϕ (of course, the rest of physics too has to satisfy the Einstein equivalence principle; the requirement that ϕ does satisfy it is only a necessary condition for consistency with general relativity).

Moreover, it turns out that $\xi = 1/6$ is necessary in order to avoid the physical pathology of *massive* fields ϕ propagating *along the light cones*.

We summarize now the derivation of this result: one begins with the physical definition of Huygens' principle due to Hadamard (1952). Assume that a point-like source of (scalar) radiation emits a delta-like pulse at time $t = 0$ in $r = 0$. If at $t = t_1$ there is radiation only on the surface of the sphere with radius $r = ct_1$ and centre $r = 0$, then we say that Huygens' principle is satisfied (and that there are no tails). If instead there is radiation also at radii $r < ct_1$ (*tails*), Huygens' principle is violated.

Mathematically, the solution for a delta-like pulse is the retarded Green function $G_{\text{R}}(x', x)$ of Eq. (1.1), which satisfies

$$[g^{a'b'}(x')\nabla_{a'}\nabla_{b'} - m^2 - \xi R(x')]G_{\text{R}}(x', x) = -\delta(x', x), \tag{2.5}$$

where $\delta(x', x)$ is the four-dimensional Dirac delta (DeWitt and Brehme, 1960) which satisfies the boundary condition $G_{\text{R}}(x', x) = 0$ if x' is in the past of x and, for simplicity, we consider the case in which the potential $V(\phi)$ reduces to a mass term (the generalization to arbitrary potentials can be found in Faraoni and Sonego (1994) and Sonego and Faraoni (1993)). $G_{\text{R}}(x', x)$ has the general structure (DeWitt and Brehme, 1960; Friedlander, 1975; Hadamard, 1952)

$$G_{\text{R}}(x', x) = \Sigma(x', x)\delta_{\text{R}}(\Gamma(x', x)) + W(x', x)\Theta_{\text{R}}(-\Gamma(x', x)), \tag{2.6}$$

where $\Gamma(x', x)$ is the square of the geodesic distance between x' and x (a quantity well known in quantum field theory in curved spaces, Birrell and Davies, 1980a); one has $\Gamma = 0$ if x' and x are light-like related, $\Gamma > 0$ if x' and x are space-like related, and $\Gamma < 0$ if x' and x are time-like related. δ_{R} is the Dirac delta with support in the past of x', and Θ_{R} is the Heaviside step function with support in the past light cone. The term in $\delta_{\text{R}}(\Gamma)$ describes a contribution to the Green function from ϕ waves propagating along the light cone ($\Gamma = 0$), while the term $\Theta_{\text{R}}(-\Gamma)$ describes the contribution to G_{R} from tails of ϕ propagating *inside* the light cone ($\Gamma < 0$). The functions $\Sigma(x', x)$ and $W(x', x)$ are mere coefficients which (at least in principle) are determined once the spacetime metric is fixed (DeWitt and Brehme, 1960; Friedlander, 1975).

The Einstein equivalence principle is imposed as follows on the physics of the field ϕ: when the spacetime manifold is progressively approximated by its tangent space (i.e., by fixing the point x and considering a small neighborhood of points x' such that $x' \to x$), then the solution $G_{\mathrm{R}}(x', x)$ for a delta-like pulse must reduce to the corresponding one known from Minkowski spacetime, which is

$$G_{\mathrm{R}}^{(\mathrm{Minkowski})}(x', x) = \frac{1}{4\pi}\delta_{\mathrm{R}}(-\Gamma) - \frac{m^2}{8\pi}\Theta_{\mathrm{R}}(\Gamma). \tag{2.7}$$

Calculations show that (Faraoni and Sonego, 1994; Sonego and Faraoni, 1993)

$$\lim\nolimits_{x'\to x} \Sigma(x', x) = \frac{1}{4\pi}, \tag{2.8}$$

$$\lim\nolimits_{x'\to x} W(x', x) = -\frac{1}{8\pi}\left[m^2 + \left(\xi - \frac{1}{6}\right) R(x)\right]; \tag{2.9}$$

hence $G_{\mathrm{R}} \to G_{\mathrm{R}}^{(\mathrm{Minkowski})}$ if and only if

$$\left(\xi - \frac{1}{6}\right) R(x) = 0, \tag{2.10}$$

and this condition is verified, in general, only if $\xi = 1/6$. Note that, if $\xi \neq 1/6$, a physical pathology may occur: the ϕ-field can have an arbitrarily large mass and still propagate along the light cone at the spacetime points where Eq. (2.10) is satisfied; in this situation an arbitrarily massive field would have no tails. It is even possible to construct an "ultrapathological" de Sitter spacetime in which the value of the constant curvature and of the mass are adjusted in such a way that a scalar field with arbitrarily large mass propagates along the light cone at every point (Faraoni and Gunzig, 1998).

The result that $\xi = 1/6$ in general relativity is extended to all metric theories of gravity in which ϕ is not part of the gravitational sector (Faraoni and Sonego, 1994; Sonego and Faraoni, 1993); in fact, in these theories, the Einstein equivalence principle holds (Will, 1993); the fact that $\xi = 1/6$ was confirmed in later studies (Grib and Poberii, 1995; Grib and Rodrigues, 1995).

3. INFLATION AND $\xi \neq 0$: THE UNPERTURBED UNIVERSE

In this section we summarize the consequences of the inclusion of NMC in the equations of the unperturbed Friedmann–Lemaitre–Robertson–Walker (FLRW) universe. We assume that the metric is given by

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \tag{3.1}$$

in comoving coordinates (t, x, y, z).

It is clear from the previous section that one cannot arbitrarily set $\xi = 0$ and it was shown in several papers (Abbott, 1981; Amendola *et al.*, 1990; Faraoni, 1996, 1998; Futamase and Maeda, 1989) that the value of ξ determines the viability of inflationary scenarios. The analysis of some specific inflationary scenarios was performed in Faraoni (1996) and is not repeated here: it suffices to mention that a scenario should be examined with regard to

(i) theoretical consistence;
(ii) fine-tuning problems.

Regarding the former, one asks oneself whether any prescription for the value of ξ is applicable. If the answer is affirmative, one examines the consequences for the viability of the specific scenario (does the value of ξ used correspond to the theoretical prescription?). Aspects studied include the existence of inflationary solutions and a sufficient amount of inflation.

Fine-tuning is an aspect perhaps less fundamental but nevertheless important; the classic example is the already mentioned chaotic inflationary scenario with $V = \lambda\phi^4$ studied by Futamase and Maeda (1989); inflationary solutions turn out to be fine-tuned for $\xi \geq 10^{-3}$, in particular for the value $\xi = 1/6 \simeq 0.16$ predicted by general relativity.

3.1. Necessary Conditions for Inflation

In this section we study necessary conditions for inflation, defined as acceleration of the scale factor, $\ddot{a} > 0$. Acceleration of the universe, the essential qualitative feature of inflation, is also required at the present epoch of the history of the universe in order to explain the data from high redshift Type Ia supernovae (Perlmutter *et al.*, 1998; Riess *et al.*, 1998). The latter imply that a form of matter with negative pressure ("quintessence") is beginning to dominate the dynamics of the universe. Scalar fields have been proposed as natural models of quintessence (Baccigalupi *et al.*, 2000; Chiba, 1999; Perrotta *et al.*, 1999; Steinhardt *et al.*, 1999; Uzan, 1999; Zlatev *et al.*, 1999), and therefore, the considerations of this subsection are also relevant for scalar field models of quintessence.

In ordinary inflation driven by a scalar field the Einstein–Friedmann equations

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa}{3}\left(\frac{\dot{\phi}^2}{2} + V(\phi)\right), \tag{3.2}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{3}(\dot{\phi}^2 - V), \tag{3.3}$$

imply that a necessary (but not sufficient) condition for cosmic acceleration is $V \geq 0$ (note that in slow-roll inflation $\rho \simeq V(\phi) \gg \dot{\phi}^2/2$ and in this case $V \geq 0$ is necessary to satisfy the weak energy condition, Wald, 1984).

What is the analog of the necessary condition for inflation when $\xi \neq 0$? Manipulation of the equations of inflation with NMC (Faraoni, 2000a) yields

$$V - \frac{3\xi}{2}\phi \frac{dV}{d\phi} > 0 \qquad \left(\xi \leq \frac{1}{6}\right). \tag{3.4}$$

This necessary condition could not be generalized to values $\xi > 1/6$, due to the difficulty of handling the dynamical equations analytically when $\xi \neq 0$ (no approximation was made). Albeit limited, the semiinfinite range of values of the coupling constant $\xi \leq 1/6$ covers many of the prescriptions for the value of ξ given in the literature. In the $\xi \to 0$ limit, Eq. (3.4) reduces to the well known necessary condition for acceleration $V > 0$.

The necessary condition (3.4) immediately allows one to reach certain conclusions.

(i) Consider an even potential $V(\phi) = V(-\phi)$ which is increasing for $\phi > 0$ (e.g., a pure mass term $m^2\phi^2/2$, a quartic potential, or their combination $V(\phi) = m^2\phi^2/2 + \lambda\phi^4 + \Lambda/\kappa$. For $0 < \xi < 1/6$, one has $\xi\phi\, dV/d\phi > 0$ and it is harder to satisfy the necessary condition (3.4) for inflation than in the minimal coupling case. Hence one can say that, for this class of potentials, it is harder to achieve acceleration of the universe, and hence inflation. If instead $\xi < 0$, the necessary condition for cosmic acceleration is more easily satisfied than in the $\xi = 0$ case, but one is not entitled to say that with NMC it is easier to achieve inflation (because a necessary, and not a sufficient condition for acceleration is considered).

(ii) Taking to the extremes the possibility of a balance between the potential $V(\phi)$ and the term $\xi R\phi^2/2$ in the action (2.1), one may wonder whether it is possible to obtain inflation with a scalar field and $V(\phi) = 0$ (i.e., a free, massless scalar with no cosmological constant, only owing to the NMC. In particular, the case of strong coupling $|\xi| \gg 1$ considered many times in the literature (Bassett and Liberati, 1998; Chiba, 1999; Fakir and Unruh, 1990b; Fukuyama *et al.*, 1997; Hwang and Noh, 1998; Morikawa, 1990, 1991; Salopek *et al.*, 1989) is of interest. It is immediate to conclude that this is not possible for $\xi \leq 1/6$ since by setting $V = 0$ the necessary condition (3.4) cannot be satisfied.

3.2. The Effective Equation of State

The effective equation of state

$$P = w\rho \tag{3.5}$$

of the cosmic fluid describing the scalar field has a coefficient w that, in general, is time-dependent; it cannot be assigned *a priori* without restricting the solutions to special ones ($a(t) = a_0 t^{2/(3(w+1))}$) if $w \neq -1$, or $a = a_0\, e^{Ht}$ if $w = -1$ (solutions for a nonspatially flat universe and arbitrary values of w can be found in Faraoni (1999). The function $w(t)$ depends on the particular solution of the equations of motion.

In the case of minimal coupling and for a *general* potential V, the effective equation of state of the universe is given by

$$\frac{P}{\rho} = \frac{\dot{\phi}^2 - 2V}{\dot{\phi}^2 + 2V} \equiv w(x), \tag{3.6}$$

where $x \equiv \dot{\phi}^2/2V$ is the ratio between the kinetic and the potential energy densities of the scalar ϕ. Under the usual assumption $V \geq 0$ (which guarantees that the energy density ρ is nonnegative when $\dot{\phi} = 0$), one has that, for $x \geq 0$, the function $w(x) = (x^2 - 1)(x^2 + 1)^{-1}$ increases monotonically from its minimum $w_{\mathrm{min}} = -1$ attained at $x = 0$ to the horizontal asymptote $+1$ as $x \to +\infty$. The slow rollover regime corresponds to the region $|x| \ll 1$ and to w near its minimum, where the kinetic energy density of ϕ is negligible in comparison to its potential energy density. As the kinetic energy density $\dot{\phi}^2/2$ increases, the equation of state progressively deviates from $P = -\rho$ and the pressure becomes less and less negative; the system gradually moves away from the slow rollover regime. At the equipartition between the kinetic and the potential energy densities ($x = 1$), one has the "dust" equation of state $P = 0$. The pressure becomes positive as x increases and, when the kinetic energy density completely dominates the potential energy density ($x \gg 1$), one finally reaches the equation of state $P = \rho$.

The limitation $-1 \leq w(x) \leq 1$ valid for $\xi = 0$ does not hold for $\xi \neq 0$: in the presence of NMC the ratio P/ρ is not bounded from below. An example is given by the exact solution with $P = -5\rho/3$ obtained in Rocha Filho *et al.* (2000) for $\xi = 1/6$ and corresponding to integrability of the equations of motion.

4. GENERALIZED SLOW-ROLL INFLATION

The equations or ordinary inflation are solved in the slow-roll approximation; similarly, the equations for the density and gravitational wave perturbations generated during inflation can only be solved, in general, in the same approximation.[9] Here, the basics of the slow-roll approximation to ordinary inflation are recalled, referring the reader to the review paper (Lidsey *et al.*, 1997) and to the references therein for a comprehensive discussion.

[9] The only exceptions are two specific scenarios: power-law inflation and the string-inspired scenario of Easther (1996), a toy model already ruled out by the COBE observations (Easther, 1996). Otherwise one may resort to numerical integration in a specified scenario.

In the approximation

$$\ddot{\phi} \ll H\dot{\phi}, \tag{4.1}$$

$$V(\phi) \approx \rho \gg \frac{\dot{\phi}^2}{2}, \tag{4.2}$$

the equations of ordinary inflation (3.2), (3.3), and

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0, \tag{4.3}$$

simplify to

$$H^2 \simeq \frac{\kappa}{3} V(\phi), \tag{4.4}$$

$$3H\dot{\phi} + \frac{dV}{d\phi} \simeq 0. \tag{4.5}$$

In this approximation, the equation of state of the cosmic fluid describing the scalar field is close to the vacuum equation of state $P = -\rho$, and the cosmic expansion is almost a de Sitter one,

$$a = a_0 \ \exp[H(t)t], \tag{4.6}$$

with

$$H(t) = H_0 + H_1 t + \cdots, \tag{4.7}$$

where H_0 is a constant and dominates the (small) term $H_1 t$ and the next orders in the expansion (4.7) of $H(t)$. The possibility that the kinetic energy density $\dot{\phi}^2/2$ of the inflaton be negligible in comparison with the potential energy density $V(\phi)$ (as expressed by Eq. (4.2)) arises if $V(\phi)$ has a flat section over which ϕ can roll slowly (i.e., with small "speed" $\dot{\phi}$). This is a necessary, but not sufficient, condition for slow-roll inflation to occur: if $V(\phi)$ is too steep, the inflaton will certainly roll fast down the potential. Indeed, the slow-roll approximation is an assumption *on the solutions* of the full equations of inflation (3.2), (3.3), and (4.3). As a matter of fact, the potential could have a flat section and ϕ could still shoot across it with large speed $\dot{\phi}$. Moreover, it was noted (Liddle, 1999) that the reduced equations of slow-roll inflation (4.4) and (4.5) have degree reduced by one in comparison with the full equations of ordinary inflation (3.2), (3.3), and (4.3). Hence, the solution is specified by the reduced set of two initial conditions $(\phi(t_0), a(t_0))$ instead of the full set of four conditions $(\phi(t_0), \dot{\phi}(t_0), a(t_0), \dot{a}(t_0))$, with an apparent loss of generality of the solutions. Then, why does the slow-roll approximation work? How is it possible that solutions of slow-roll inflation be general solutions? (If they correspond to a set of zero measure in the set of all initial conditions, they are fine-tuned and clearly unphysical.) The answer is that the de Sitter solutions

$$a = a_0 \exp(H_0 t), \qquad \phi = 0 \tag{4.8}$$

are *attractor points* for the orbits of the solutions in the phase space (Liddle *et al.*, 1994; Salopek and Bond, 1990). Therefore, the quasi-exponential expansion (4.6) is a *general* property. Were inflationary (de Sitter) attractors absent, slow-roll inflation would be an empty theory without generic solutions, a formalism describing a speculation that doesn't occur in the real world.

How does the attractor mechanism transfer to the case of generalized inflation? Does the attractor property of de Sitter solutions survive when NMC is included in the picture? Is a flat section of the potential still a necessary condition for slow-roll inflation? Regarding the last question, it is useful to keep in mind that (as has been known for a long time, Abbott, 1981; Faraoni, 1996; Futamase and Maeda, 1989) the NMC term $\xi R\phi^2/2$ in the action (2.1) acts as an effective mass term,[10] spoiling the flatness of the potentials that are known to be inflationary for $\xi = 0$. These considerations will be reexamined and made quantitative in the following.

One begins the analysis by writing the equations of generalized inflation as

$$6[1 - \xi(1 - 6\xi)\kappa\phi^2](\dot{H} + 2H^2) - \kappa(6\xi - 1)\dot{\phi}^2 - 4\kappa V + 6\kappa\xi\phi V' = 0, \tag{4.9}$$

$$\frac{\kappa}{2}\dot{\phi}^2 + 6\xi\kappa H\phi\dot{\phi} - 3H^2(1 - \kappa\xi\phi^2) + \kappa V = 0, \tag{4.10}$$

$$\ddot{\phi} + 3H\dot{\phi} + \xi R\phi + V' = 0. \tag{4.11}$$

Eqs. (4.9)–(4.11) are derived by varying the action (2.1); Eq. (4.9) corresponds to the trace of the Einstein equations, $R = -\kappa(\rho - 3P)$; Eq. (4.10) is the Hamiltonian constraint $3H^2 = \kappa\rho$, while Eq. (4.11) is the well-known Klein–Gordon equation (1.1).

Note that, in the presence of NMC, the energy–momentum tensor of the scalar field, and consequently its energy density ρ and pressure P can be identified in several possible inequivalent ways, corresponding to different ways of writing the field equations (see Faraoni, 2000a for a detailed discussion). The procedure that we adopt is identified as a convenient one in Faraoni (2000a) because it is general (i.e., solutions are not lost by manipulation of the field equations) and the stress–energy tensor of the scalar field is covariantly conserved, which may not happen for other choices of ρ and P (Faraoni, 2000a).

Explicitly, the energy density and pressure of ϕ (which we assume to be the only source of gravity during inflation) relative to a comoving observer of the

[10] Although the effect is like that of a mass, the interpretation of the constant curvature as a mass term for the scalar field must not be taken literally (Faraoni and Cooperstock, 1998).

FLRW universe are given by

$$\rho = \frac{\dot{\phi}^2}{2} + 3\xi H^2\phi^2 + 6\xi H\phi\dot{\phi} + V(\phi), \tag{4.12}$$

$$P = \frac{\dot{\phi}^2}{2} - V(\phi) - \xi(4H\phi\dot{\phi} + 2\dot{\phi}^2 + 2\phi\ddot{\phi}) - \xi(2\dot{H} + 3H^2)\,\phi^2. \tag{4.13}$$

As discussed in detail in Gunzig *et al.* (2000, in press), only two equations of the set (4.9)–(4.11) are independent, and the system can be reduced to a two-dimensional phase space manifold with variables (H, ϕ). It is then straightforward to verify that the solutions

$$(H, \phi) = (H_0, \phi_0), \tag{4.14}$$

with H_0 and ϕ_0 constants, are all the fixed points of the dynamical system with $\xi \neq 0$, provided that the conditions

$$12\xi H_0^2\phi_0 + V_0' = 0, \tag{4.15}$$

$$H_0^2\left(1 - \kappa\xi\phi_0^2\right) = \frac{\kappa V_0}{3}, \tag{4.16}$$

are satisfied, where $V_0 \equiv V(\phi_0)$ and $V_0' \equiv dV/d\phi|_{\phi 0}$. There are only two such constraints since only two equations in the set (4.9)–(4.11) are independent. The fixed points (4.14) are de Sitter solutions with constant scalar field and generalize the solutions $(H, \phi) = (\pm\sqrt{\Lambda/3}, 0)$ well known for minimal coupling, $\Lambda > 0$ being the cosmological constant (corresponding to the constant potential $V = \Lambda/\kappa$).

In order to assess the stability of the universes (4.14) (i.e., to decide whether they are attractors or not), one has to perform a stability analysis with respect to perturbations $\delta\phi$ and δH of the scalar field and the Hubble parameter,[11]

$$\phi(t, \vec{x}) = \phi_0(t) + \delta\phi(t, \vec{x}), \qquad H(t, \vec{x}) = H_0 + \delta H(t, \vec{x}). \tag{4.17}$$

Since the general perturbations are space- and time-dependent, one is faced with the recurrent problem of gauge-dependence in cosmology: if the perturbation analysis is performed in a particular gauge (of which many appear in the literature), one can never be sure that the growing (decaying) modes are genuine perturbations and not pure gauge modes which can be removed by coordinate transformations (Kolb and Turner, 1994; Linde, 1990).

To solve the problem, one needs to perform a gauge-independent analysis: we adopt the covariant and gauge-invariant formalism of Bardeen (1980), in the modified formulation of Ellis *et al.* (1989, 1990), Ellis and Bruni (1989), Hwang

[11] In the analysis of the phase space, attention is usually restricted only to time-dependent perturbations (e.g., Halliwell, 1987; Muller *et al.*, 1990); however, these perturbations are too special to draw definite conclusions.

and Vishniac (1990), and Mukhanov *et al.* (1992). We first present and discuss the results (Faraoni, 2000b), postponing their derivation to the final part of this section. For *expanding* de Sitter spaces (4.14) with $H_0 > 0$, there is *stability* (and therefore (4.14) is an attractor point) if

$$V_0'' \geq \frac{V_0'}{\phi_0} \frac{1 - 3\xi\kappa\phi_0^2}{1 - \xi\kappa\phi_0^2} \qquad (\phi_0 \neq 0), \tag{4.18}$$

$$V_0'' + 4\xi\kappa V_0 \geq 0 \qquad (\phi_0 = 0). \tag{4.19}$$

By contrast, the *contracting* fixed points (4.14) with $H_0 < 0$ are always unstable, like in the case of minimal coupling.

Stability depends not only on the form of the scalar field potential, which is expected, but also on the value of ξ. It is only in particular situations that the ξ-dependence disappears and stability holds irrespective of the value of ξ. This happens, for example

(i) if $V(\phi)$ has a minimum ($V_0' = 0$ and $V_0'' > 0$) at ϕ_0;
(ii) if $V = \Lambda/\kappa + \lambda\phi^n$ (including the case of a simple mass term $m^2\phi^2/2$) with $\Lambda, \lambda \geq 0$.

This space is stable for $n \geq 1 + f(x)$, where

$$x \equiv \kappa\xi\phi_0^2, \qquad f(x) = \frac{1 - 3x}{1 - x} < 1. \tag{4.20}$$

The stability conditions (4.18) and (4.19) are deduced by assuming that $0 < x < 1$; if $x > 1$ a negative effective gravitational coupling $G_{\text{eff}} \equiv G(1 - \kappa\xi\phi_0^2)^{-1}$ arises (Faraoni, 2000a; Futamase and Maeda, 1989). Furthermore, the slow-roll parameter ϵ_3 defined in the next subsection diverges if the unperturbed solution $\phi(t)$ crosses one of the critical values

$$\pm\phi_1 \equiv \pm\frac{1}{\sqrt{\kappa\xi}} \tag{4.21}$$

(which are defined for $\xi > 0$), while the slow-roll parameter ϵ_4 diverges if $\phi(t)$ crosses one of the other critical values

$$\pm\phi_2 \equiv \pm\frac{1}{\sqrt{\kappa\xi(1 - 6\xi)}} \tag{4.22}$$

(which exist for $0 < \xi < 1/6$).

Under the usual assumption that V be nonnegative, the Hamiltonian constraint (4.10) forces $|\phi|$ to be smaller than ϕ_2 (Amendola *et al.*, 1990; Futamase and Maeda, 1989); we further assume that $|\phi| < \phi_1$. If instead $|\phi| > \phi_1$, the direction of the inequality (4.18) is reversed.

The case $\phi = \pm\phi_1$ not considered so far corresponds to a class of solutions with constant Ricci curvature containing a de Sitter representative (Gunzig *et al.*, 2000; Gunzig *et al.*, in press). However, the latter is clearly fine-tuned and unstable with respect to perturbations $\Delta\phi$.

For $\xi = 0$, Eq. (4.16) yields $V_0' = 0$ for the fixed point, whereas the stability condition (4.18) gives $V_0'' > 0$; this happens, for example, when $V(\phi)$ has a minimum Λ/κ in ϕ_0, which intuitively corresponds to stability. A solution starting at any value of ϕ is attracted toward the minimum; if ϕ identically coincides with ϕ_0 and there is no kinetic energy ($\dot{\phi} = 0$), Eqs. (4.12) and (4.13) yield the energy density $\rho = 3\xi H_0^2 \phi_0^2 + V_0 = -P$ and the vacuum equation of state (corresponding to de Sitter solutions) holds.

If instead $V_0'' < 0$ and the potential has a maximum $V_0 = \Lambda/\kappa$ in ϕ_0, a solution starting near ϕ_0 will run away from it.

When $\xi \neq 0$ the interpretation of the stability conditions (4.18) and (4.19) is complicated by the balance between $V(\phi)$ and $\xi R\phi^2/2$ in the action (2.1). Eqs. (4.18) and (4.19) make precise the previous qualitative considerations on this balance in Abbott (1981), Fakir *et al.* (1992), Faraoni (1996), and Futamase and Maeda (1989).

As a conclusion, *slow-roll inflation only makes sense for $\xi \neq 0$ when the conditions* (4.18) *or* (4.19) *are satisfied*. In this case, the expanding de Sitter spaces (4.14) satisfying Eqs. (4.15) and (4.16) are attractor points in the phase space. One must be cautious and check that Eq. (4.18) or (4.19) are satisfied before applying the slow-roll formalism presented in the next section. The importance of the inflationary attractors is made clear once again by the example of the *contracting* spaces (4.14), for which the slow-roll approximation is *exact* (in the sense that the slow-roll parameters ϵ_i defined in the next section vanish exactly). However, this bears no relationship with the actual inflationary solutions because the contracting spaces (4.14) are not attractors.

4.1. Derivation of the Stability Conditions

The derivation of Eqs. (4.18) and (4.19) proceeds as follows: the metric perturbations are identified by the quantities A, B, H_L, and H_T in the expression of the spacetime metric

$$ds^2 = a^2(t)\{-(1 + 2AY)\,dt^2 - 2BY_i\,dt\,dx^i + [\delta_{ij}(1 + 2H_L) + 2H_T Y_{ij}]\,dx^i\,dx^j\}, \tag{4.23}$$

where the Y are scalar harmonics satisfying

$$\nabla^2 Y = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) Y = -k^2 Y, \tag{4.24}$$

Y_i and Y_{ij} are related to the derivatives of the Y by

$$Y_i = \frac{1}{k^2}\partial_i Y, \tag{4.25}$$

$$Y_{ij} = \frac{1}{k^2}\partial_i\partial_j Y + \frac{1}{3}\delta_{ij} Y, \tag{4.26}$$

respectively (Bardeen, 1980), and k is the eigenvalue defined by Eq. (4.24). We shall use Bardeen's gauge-invariant potentials

$$\Phi_H = H_L + \frac{\dot{a}}{k}\left(B - \frac{a}{k}\dot{H}_T\right), \tag{4.27}$$

$$\Phi_A = A + \frac{\dot{a}}{k}\left(B - \frac{a}{k}\dot{H}_T\right) + \frac{a}{k}\left[\dot{B} - \frac{1}{k}(a\dot{H}_T)^{\cdot}\right], \tag{4.28}$$

and the Ellis–Bruni–Hwang (Ellis *et al.*, 1989, 1990; Ellis and Bruni, 1989) variables

$$\begin{aligned} \Delta\phi(t,\vec{x}) &= \delta\phi + \frac{a}{k}\dot{\phi}\left(B - \frac{a}{k}\dot{H}_T\right), \\ \Delta R(t,\vec{x}) &= \delta R + \frac{a}{k}\dot{R}\left(B - \frac{a}{k}\dot{H}_T\right). \end{aligned} \tag{4.29}$$

The evolution equations for the gauge-invariant variables $\Phi_{H,A}$ and $\Delta\phi$ were derived in Hwang (1990a):

$$\begin{aligned} &\dot{\Phi}_H + \left(\frac{\xi\kappa\phi\dot{\phi}}{1-\kappa\xi\phi^2} - H\right)\Phi_A - \frac{\kappa}{1-\kappa\xi\phi^2} \\ &\quad\times\left\{\xi\phi\Delta\dot{\phi} + \left[\xi\phi\left(\frac{\dot{\phi}}{\phi} - H\right) - \frac{\dot{\phi}}{2}\right]\Delta\phi\right\} = 0, \end{aligned} \tag{4.30}$$

$$\begin{aligned} &\left(\frac{k}{a}\right)^2\Phi_H + \frac{1}{1-\kappa\xi\phi^2}\left(\frac{3\xi^2\kappa\phi^2}{1-\kappa\xi\phi^2} + \frac{1}{2}\right)\kappa\dot{\phi}^2\Phi_A \\ &- \frac{1}{1-\kappa\xi\phi^2}\left\{\left(\frac{3\xi^2\kappa\phi^2}{1-\kappa\xi\phi^2} + \frac{1}{2}\right)\kappa\dot{\phi}\Delta\dot{\phi}\right. \\ &\left. + \left[\left(\frac{k}{a}\right)^2\xi\phi - \ddot{\phi}\left(\frac{3\xi^2\kappa\phi^2}{1-\kappa\xi\phi^2} + \frac{1}{2}\right)\right]\kappa\Delta\phi\right\} = 0, \end{aligned} \tag{4.31}$$

$$\Phi_A + \Phi_H - \frac{2\xi\kappa\phi\Delta\phi}{1-\kappa\xi\phi^2} = 0, \tag{4.32}$$

$$\ddot{\Phi}_H + H\dot{\Phi}_H + \left(H - \frac{\xi\kappa\phi\dot{\phi}}{1-\kappa\xi\phi^2}\right)(2\dot{\Phi}_H - \dot{\Phi}_A) - \frac{\kappa V}{1-\kappa\xi\phi^2}\Phi_A$$
$$+ \frac{\kappa}{1-\kappa\xi\phi^2}\left\{-\xi\phi\Delta\ddot{\phi} + \left[\frac{\dot{\phi}}{2} - 2\xi(\dot{\phi} + H\phi)\right]\Delta\dot{\phi}\right.$$
$$\left. + \left[\xi\phi\left(\kappa p_H - \frac{\ddot{\phi}}{\phi} - \frac{2H\dot{\phi}}{\phi}\right) - \frac{V'}{2\kappa}\right]\kappa\Delta\phi\right\} = 0, \qquad (4.33)$$

$$\Delta\ddot{\phi} + 3H\Delta\dot{\phi} + \left(\frac{k^2}{a^2} + \xi R + V''\right)\Delta\phi + \dot{\phi}(3\dot{\Phi}_H - \dot{\Phi}_A)$$
$$+ 2(V' + \xi R\phi)\Phi_A + \xi\phi\Delta R = 0, \qquad (4.34)$$

where

$$p_H = \frac{1}{1-\kappa\xi\phi^2}\left[\frac{\dot{\phi}^2}{2} - V - 2\xi\phi\left(\ddot{\phi} + 3H\dot{\phi} + \frac{\dot{\phi}^2}{\phi}\right)\right]. \qquad (4.35)$$

An overdot denotes differentiation with respect to the comoving time of the unperturbed background, and the subscript zero denotes unperturbed quantities. The formulation of Hwang (1990a) has been adapted to the case of a FLRW universe with flat spatial sections; the constant $\kappa = 8\pi G$ has been restored. Only first order calculations in the perturbations are presented here.

Considerable simplifications occur in Eqs. (4.30)–(4.35) for the case of a de Sitter space with constant scalar field (4.14) as the background universe; to first order, one obtains

$$\Phi_H = \Phi_A = \frac{\xi\kappa\phi_0}{1-\kappa\xi\phi_0^2}\Delta\phi, \qquad (4.36)$$

$$\Delta\ddot{\phi} + 3H_0\Delta\dot{\phi} + \left[\frac{k^2}{a^2} + V_0'' + \frac{\xi R_0\left(1+\kappa\xi\phi_0^2\right) + 2V_0'\kappa\xi\phi_0}{1-\kappa\xi\phi_0^2}\right]\Delta\phi + \xi\phi_0\Delta R = 0, \qquad (4.37)$$

while Eq. (4.33) reduces to the constraint

$$-\frac{V_0\xi\phi_0}{1-\kappa\xi\phi_0^2} = V_0' - p_H\xi\phi_0 \qquad (4.38)$$

which, using Eq. (4.35), is written as

$$\frac{V_0'}{V_0} = -\frac{4\kappa\xi\phi_0}{1-\kappa\xi\phi_0^2}; \qquad (4.39)$$

Eq. (4.39) can also be obtained by division of Eq. (4.15) by Eq. (4.16). Using the fact that $R = 6(\dot{H} + 2H^2)$ and Eq. (4.15) one obtains

$$\Delta\ddot{\phi} + 3H_0\Delta\dot{\phi} + \left(V_0'' + 12\xi H_0^2 + \frac{k^2}{a^2}\right)\Delta\phi + \xi\phi_0\Delta R = 0. \tag{4.40}$$

For a de Sitter background (4.14) the gauge-invariant variables $\Delta\phi$ and ΔR coincide, respectively, with the scalar field and curvature perturbations $\delta\phi$ and δR, to first order,

$$\Delta\phi = \delta\phi, \qquad \Delta R = \delta R = 6(\delta\dot{H} + 4H_0\delta H), \tag{4.41}$$

and therefore[12]

$$\Delta R = \delta R = \frac{-6\xi\kappa\phi_0\left[V_0'' + 4(1+3\xi)H_0^2\right]}{1 - \xi(1-6\xi)\kappa\phi_0^2}\Delta\phi. \tag{4.42}$$

One can then substitute Eq. (4.42) into Eq. (4.40) for $\Delta\phi$ and use Eq. (4.15) to obtain

$$\Delta\ddot{\phi} + 3H_0\Delta\dot{\phi} + \left(\frac{k^2}{a^2} + \alpha\right)\Delta\phi = 0, \tag{4.43}$$

where

$$\alpha = \frac{V_0''\phi_0\left(1 - \kappa\xi\phi_0^2\right) - V_0'\left(1 - 3\kappa\xi\phi_0^2\right)}{\phi_0\left[1 - \xi(1-6\xi)\kappa\phi_0^2\right]} \tag{4.44}$$

and $a = a_0 \exp(H_0 t)$. Let us consider the expanding ($H_0 > 0$) de Sitter spaces (4.14): at late times $t \to +\infty$ one can neglect the $(k/a)^2 \propto e^{-2H_0 t}$ term in Eq. (4.43) and look for solutions of the form

$$\Delta\phi(t, \vec{x}) = \frac{1}{(2\pi)^{3/2}}\int d^3\vec{l}\,\Delta\phi_l(t)\, e^{i\vec{l}\cdot\vec{x}}, \quad \Delta\phi_l(t) = \epsilon_l\, e^{\beta_l t}. \tag{4.45}$$

Note that the Fourier expansion (4.45) is well defined because the universe has flat spatial sections. The constants β_l must satisfy the algebraic equation

$$\beta_l^2 + 3H_0\beta_l + \alpha = 0, \tag{4.46}$$

with roots

$$\beta_l^{(\pm)} = \frac{3H_0}{2}\left(-1 \pm \sqrt{1 - \frac{4\alpha}{9H_0^2}}\right). \tag{4.47}$$

While $Re(\beta_l^{(-)}) < 0$, the sign of $Re(\beta_l^{(+)})$ depends on α: $Re(\beta_l^{(+)}) > 0$ if $\alpha < 0$ and $Re(\beta_l^{(+)}) \leq 0$ if $\alpha \geq 0$. Hence one has *stability* for $\alpha \geq 0$ which, for $\phi_0 \neq 0$ translates into the advertised result (4.18). If instead $\alpha < 0$, the gauge-invariant

[12] This expression agrees with the one following Eq. (38) of Hwang (1990a).

perturbations $\Delta\phi$ and $\Delta R \propto \Delta\phi$ (cf. Eq. (4.42)) grow without bound and there is *instability*.

Let us discuss now the $\phi_0 = 0$ case; Eqs. (4.30)–(4.34) yield

$$\Phi_H = \Phi_A = 0, \tag{4.48}$$

$$\Delta\ddot{\phi} + 3H_0\Delta\dot{\phi} + \left(\frac{k^2}{a^2} + \alpha_1\right)\Delta\phi = 0, \tag{4.49}$$

where $\Delta R = 0$ and $\alpha_1 = V_0'' + 4\xi\kappa V_0$; hence, for $\phi_0 = 0$, there is *stability* if Eq. (4.19) is satisfied and instability otherwise.

Finally, consider the *contracting* ($H_0 < 0$) fixed points (4.14); in this case it is convenient to use conformal time η (defined by $dt = a\,d\eta$) and the auxiliary variable $u \equiv a\Delta\phi$. Eq. (4.43) becomes

$$\frac{d^2u}{d\eta^2} + [k^2 - U(\eta)]\,u = 0, \tag{4.50}$$

where

$$U(\eta) = \left(4 - \frac{\alpha_1}{H_0^2}\right)\frac{1}{\eta^2} + \frac{2}{H_0\eta^3}, \tag{4.51}$$

and we used the relation

$$\eta = -\frac{1}{aH_0} \tag{4.52}$$

vaild in the background (4.14) (see Appendix B). Formally, Eq. (4.50) is a one-dimensional Schrödinger equation for a quantum particle of unit mass in the potential $U(\eta)$; its asymptotic solutions at large η (corresponding to $t \rightarrow +\infty$ for a contracting de Sitter background) are free waves $u \simeq e^{\pm ik\eta}$, and $\Delta\phi \propto H_0\eta$ diverges. The solutions (4.14) with $H_0 < 0$ are *unstable*, as in the $\xi = 0$ case.

4.2. Slow-Roll Parameters

The Hubble slow-roll approximation known for ordinary inflation (Lidsey *et al.*, 1997 and references therein) is characterized by two slow-roll parameters $\epsilon_H = -\dot{H}/H^2$ and $\eta_H = -\ddot{\phi}/(H\dot{\phi})$ which stay small during slow-roll inflation. When ϵ_H and η_H increase, the kinetic energy of the inflation increases and, when ϵ_H and η_H become of order unity, the slow-roll approximation breaks down and inflation ends.

Slow-roll parameters have been identified also for generalized inflation (Hwang, 1990a; Hwang and Noh, 1996; Kaiser, 1995a,b); the novelty is that there are four such parameters as opposed to the two of ordinary inflation. From the point of view of Section 3, this fact may provide a rationale of why it is harder to achieve slow-roll inflation with nonminimal rather than minimal coupling, for a given potential $V(\phi)$ one has to satisfy four slow-roll necessary conditions instead of two. The slow-roll parameters are the dimensionless quantities

$$\epsilon_1 = \frac{\dot{H}}{H^2} = -\epsilon_H, \tag{4.53}$$

$$\epsilon_2 = \frac{\ddot{\phi}}{H\dot{\phi}} = -\eta_H, \tag{4.54}$$

$$\epsilon_3 = -\frac{\xi\kappa\phi\dot{\phi}}{H[1-(\phi/\phi_1)^2]}, \tag{4.55}$$

$$\epsilon_4 = -\frac{\xi(1-6\xi)\kappa\phi\dot{\phi}}{H[1-(\phi/\phi_2)^2]}. \tag{4.56}$$

ϵ_3 and ϵ_4 vanish in the limit $\xi \to 0$ of ordinary inflation; ϵ_4 also vanishes for conformal coupling ($\xi = 1/6$). One has $|\epsilon_i| \ll 1$ for every solution attracted by the expanding de Sitter spaces (4.14) (when the latter are attractor points) at sufficiently large times. Moreover, $\epsilon_i = 0$ exactly for de Sitter solutions.

5. INFLATION AND $\xi \neq 0$: PERTURBATIONS

The quantum fluctuations of the inflaton field which unavoidably take place during inflation generate density (scalar) perturbations that act as seeds for the formation of the structures observed in the universe today, from galaxies to superclusters (Kolb and Turner, 1994; Linde, 1990). Similarly, quantum fluctuations δg_{ab} of the metric tensor are generated during inflation, corresponding to gravitational waves (Kolb and Turner, 1994; Linde, 1990). Both scalar and tensor perturbations leave an imprint on the cosmic microwave background by generating temperature fluctuations. The latter have been detected by COBE (Smoot *et al.*, 1992) and other experiments, and are going to be studied with unprecedented accuracy by the MAP[13] and PLANCK[14] satellites to be launched in the very near future.

In order to confront itself with the present and future observations, the theory must predict observables such as the amplitudes and spectra of perturbations. For ordinary inflation, these have been computed.[15] In a remarkable series of papers, Hwang (1990b, 1996, 1997a) and Hwang and Noh (1996) have performed a similar calculation for generalized gravity theories described by the action

$$S = \int d^4x\,\sqrt{-g}\left[\frac{1}{2}\,f(\phi, R) - \frac{\omega(\phi)}{2}\nabla^c\phi\nabla_c\phi - V(\phi)\right]. \tag{5.1}$$

[13] See footnote 5.
[14] See footnote 6.
[15] The calculation took a long time to be completed, starting with the early efforts of the early 80s (see Lidsey *et al.* (1997) for a recent review and Guth (1997) for a historical perspective).

The case of a nonminimally coupled scalar field is recovered by setting

$$f(\phi, R) = \frac{R}{\kappa} - \xi R\phi^2, \quad \omega = 1. \tag{5.2}$$

Hwang's treatment is covariant and gauge-invariant and builds upon the formalism developed by Bardeen (1980), Ellis *et al.* (1989, 1990), Ellis and Bruni (1989), and Hwang and Vishniac (1990), and considers a FLRW universe with arbitrary curvature index. Motivated by inflation, we restrict ourselves to the spatially flat case. The idea of Hwang (1990b, 1996, 1997a) and Hwang and Noh (1996) is to reduce the field equations of the theory to formal Einstein equations

$$G_{ab} = \kappa T_{ab}^{(\text{eff})}, \tag{5.3}$$

where $T_{ab}^{(\text{eff})}$ is an effective stress–energy tensor incorporating terms that would normally appear in the left hand side of the field equations. The treatment proceeds by using the gauge-invariant study of perturbations in Einstein gravity and ordinary inflation (Lidsey *et al.*, 1997; Mukhanov *et al.*, 1992).

In the following, we review and complete the calculation, adapting it to the case of the action (2.1) and (5.2); we believe that this review is useful for future reference, since a consistent discussion of the slow-roll approximation was not given before for generalized inflation. Instead of using Eq. (4.23), it is convenient to rewrite the metric perturbations in a different form.

5.1. Scalar Perturbations

The metric is written as

$$ds^2 = -(1+2\alpha)\,dt^2 - \chi_{,i}\,dt\,dx^i + a^2(t)\delta_{ij}(1+2\varphi)\,dx^i\,dx^j, \tag{5.4}$$

while the scalar field is given by Eq. (4.17). One introduces the additional gauge-invariant variable

$$\delta\phi_\varphi = \delta\phi - \frac{\dot\phi}{H}\varphi \equiv -\frac{\dot\phi}{H}\varphi_{\delta\phi}. \tag{5.5}$$

The second order action for the perturbations (analogous to the one for ordinary inflation, Lidsey *et al.*, 1997) is Hwang (1997b, 1998a)

$$S_{\text{pert}} = \int dt\,d^3\vec{x}\,\mathcal{L}_{\text{pert}} = \frac{1}{2}\int dt\,d^3\vec{x}\,a^3 Z\left\{\delta\dot\phi_\varphi^2 - \frac{1}{a^2}\delta\phi_\varphi^{;i}\delta\phi_{\varphi,i}\right.$$
$$\left. + \frac{1}{a^3 Z}\frac{H}{\dot\phi}\left[a^3 Z\left(\frac{\dot\phi}{H}\right)^{\cdot}\right]^{\cdot}\delta\phi_\varphi^2\right\}, \tag{5.6}$$

where

$$Z(t) = \frac{H^2[1-\kappa\xi(1-6\xi)\phi^2](1-\kappa\xi\phi^2)}{[H(1-\kappa\xi\phi^2) - \xi\kappa\phi\dot\phi]^2} \tag{5.7}$$

(cf. Eq. (6) of Hwang (1997b, 1998a) and use Eq. (5.2)). The action (5.6) yields the evolution equation for the perturbations $\delta\phi_\varphi$

$$\delta\ddot{\phi}_\varphi + \frac{(a^3Z)^{\cdot}}{a^3Z}\delta\dot{\phi}_\varphi - \left\{\frac{\nabla^2}{a^2} + \frac{1}{a^3Z}\frac{H}{\dot{\phi}}\left[a^3Z\left(\frac{\dot{\phi}}{H}\right)^{\cdot}\right]^{\cdot}\right\}\delta\phi_\varphi = 0. \tag{5.8}$$

By using the auxiliary variables[16]

$$z(t) = \frac{a\dot{\phi}}{H}\sqrt{Z}, \tag{5.9}$$

$$v(t,\vec{x}) = z\frac{H}{\dot{\phi}}\delta\phi_\varphi = a\sqrt{Z}\delta\phi_\varphi, \tag{5.10}$$

Eq. (5.8) is reduced to

$$v_{\eta\eta} - \left(\nabla^2 + \frac{z_{\eta\eta}}{z}\right)v = 0, \tag{5.11}$$

where η denotes conformal time.

Quantization is achieved by assuming that the background is classical while the perturbations have quantum nature. A Heisenberg picture is used in which quantum operators change in time while the state vectors remain constant (the vacuum state of the system is identified with the adiabatic vacuum (Birrell and Davies, 1980a), and one wants the vacuum state to remain unchanged in time). The fluctuations $\delta\phi(t,\vec{x})$ of the scalar field are associated to a quantum operator $\delta\hat{\phi}(t,\vec{x})$; similarly $\varphi \rightarrow \hat{\varphi}$, and the gauge-invariant variable (5.5) is associated to the quantum operator

$$\delta\hat{\phi}_\varphi = \delta\hat{\phi} - \frac{\dot{\phi}}{H}\hat{\varphi} \tag{5.12}$$

(the hats denote quantum operators). The unperturbed quantities are regarded as classical.

Since the three-dimensional space is flat, it is meaningful to perform a Fourier decomposition of the operator $\delta\hat{\phi}_\varphi$,

$$\delta\hat{\phi}_\varphi = \frac{1}{(2\pi)^{3/2}}\int d^3\vec{k}\,[\hat{a}_k\delta\phi_{\varphi k}(t)\,e^{i\vec{k}\cdot\vec{x}} + \hat{a}_k^\dagger\delta\phi_{\varphi k}^*(t)\,e^{-i\vec{k}\cdot\vec{x}}], \tag{5.13}$$

where the annihilation and creation operators $\hat{a}_k$ and $\hat{a}_k^\dagger$ satisfy the canonical commutation relations

$$[\hat{a}_k, \hat{a}_{k'}] = [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] = 0, \tag{5.14}$$

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta^{(3)}(\vec{k} - \vec{k}'), \tag{5.15}$$

[16] The variable z of Eq. (5.9) agrees with the z-variable of Mukhanov *et al.* (1992) and with the z of Lidsey *et al.* (1997) multiplied by the factor $\sqrt{Z}$ (note that $Z = 1$ corresponds to ordinary inflation).

and the mode functions $\delta\phi_{\varphi k}(t)$ are complex Fourier coefficients satisfying the classical equations obtained from Eq. (5.2)

$$\delta\ddot{\phi}_{\varphi k} + \frac{(a^3 Z)^{\cdot}}{a^3 Z}\delta\dot{\phi}_{\varphi k} + \left\{\frac{k^2}{a^2} - \frac{1}{a^3 Z}\frac{H}{\dot{\phi}}\left[a^3 Z\left(\frac{\dot{\phi}}{H}\right)^{\cdot}\right]\right\}\delta\phi_{\varphi k} = 0. \tag{5.16}$$

Similarly,

$$v(t, \vec{x}) = \frac{1}{(2\pi)^{3/2}}\int d^3\vec{k}\,[v_k(t)\,e^{i\vec{k}\cdot\vec{x}} + v_k^*(t)\,e^{-i\vec{k}\cdot\vec{x}}], \tag{5.17}$$

$$\hat{v} = \frac{zH}{\dot{\phi}}\delta\hat{\phi}_{\varphi} = a\sqrt{Z}\delta\hat{\phi}_{\varphi}, \tag{5.18}$$

and the $v_k(t)$ satisfy the equation

$$(v_k)_{\eta\eta} + \left(k^2 - \frac{z_{\eta\eta}}{z}\right)v_k = 0. \tag{5.19}$$

The momentum conjugated to $\delta\phi_{\varphi}$ is

$$\delta\pi_{\phi}(t, \vec{x}) = \frac{\partial\mathcal{L}_{\text{pert}}}{\partial(\delta\dot{\phi}_{\varphi})} = a^3 Z\delta\dot{\phi}_{\varphi}(t, \vec{x}), \tag{5.20}$$

and the associated quantum operator is $\delta\hat{\pi}_{\varphi}$.

$\delta\hat{\phi}_{\varphi}$ and $\delta\hat{\pi}_{\varphi}$ satisfy the equal time commutation relations

$$[\delta\hat{\phi}_{\varphi}(t, \vec{x}), \delta\hat{\phi}_{\varphi}(t, \vec{x}\,')] = [\delta\hat{\pi}_{\varphi}(t, \vec{x}), \delta\hat{\pi}_{\varphi}(t, \vec{x}\,')] = 0, \tag{5.21}$$

$$[\delta\hat{\phi}_{\varphi}(t, \vec{x}), \delta\hat{\pi}_{\varphi}(t, \vec{x}\,')] = \frac{i}{a^3 Z}\delta^{(3)}(\vec{x} - \vec{x}\,'). \tag{5.22}$$

$\delta\phi_{\varphi k}(t)$ satisfy the Wronskian condition

$$\delta\phi_{\varphi k}\delta\dot{\phi}^*_{\varphi k} - \delta\phi^*_{\varphi k}\delta\dot{\phi}_{\varphi k} = \frac{i}{a^3 Z}. \tag{5.23}$$

In Hwang (1990b, 1996, 1997a) and Hwang and Noh (1996) it is *assumed* that

$$\frac{z_{\eta\eta}}{z} = \frac{n}{\eta^2}, \tag{5.24}$$

where n is a constant; we will comment later on the validity of this assumption. Under the assumption (5.24), Eq. (5.19) for the Fourier modes v_k reduces to

$$(v_k)_{\eta\eta} + \left[k^2 - \frac{(\nu^2 - 1/4)}{\eta^2}\right]v_k = 0, \tag{5.25}$$

where

$$\nu = \left(n + \frac{1}{4}\right)^{1/2}. \tag{5.26}$$

By making the substitutions

$$s = k\eta, \qquad v_k = \sqrt{s}\,J(s), \tag{5.27}$$

Eq. (5.25) is reduced to the Bessel equation

$$\frac{d^2 J}{ds^2} + \frac{1}{s}\frac{dJ}{ds} + \left(1 - \frac{\nu^2}{s^2}\right) J = 0; \tag{5.28}$$

therefore the solutions $v_k(\eta)$ can be expressed as

$$v_k(\eta) = \sqrt{k\eta}\, J_\nu(k\eta), \tag{5.29}$$

where $J_\nu(s)$ are Bessel functions of order ν. Eq. (5.10) yields the solutions for the Fourier coefficients $\delta\phi_{\varphi k}$

$$\delta\phi_{\varphi k}(\eta) = \frac{\dot{\phi}}{zH} v_k(\eta) = \frac{1}{a\sqrt{Z}} v_k(\eta). \tag{5.30}$$

The $v_k(\eta)$ which solve the Bessel equation (5.25) are expressed in terms of Hankel functions $H_\nu^{(1,2)}$, leading to (see Appendix A)

$$v_k(\eta) = \frac{\sqrt{\pi|\eta|}}{2}\left[c_1(\vec{k})H_\nu^{(1)}(k|\eta|) + c_2(\vec{k})H_\nu^{(2)}(k|\eta|)\right] \tag{5.31}$$

and, by using Eq. (5.30), to

$$\delta\phi_{\varphi k}(\eta) = \frac{\sqrt{\pi|\eta|}}{2a\sqrt{Z}}\left[c_1(\vec{k})H_\nu^{(1)}(k|\eta|) + c_2(\vec{k})H_\nu^{(2)}(k|\eta|)\right]. \tag{5.32}$$

The normalization is chosen in such a way that the relation

$$|c_2(\vec{k})|^2 - |c_1(\vec{k})|^2 = 1 \tag{5.33}$$

holds, in order to preserve the equal time commutation relation (5.23). Furthermore, the coefficients are completely fixed by requiring that, in the limit of small scales, the vacuum corresponds to positive frequency solutions; in fact the field theory in Minkowski space must be recovered in this limit.[17] The small scale (large wavenumber) limit corresponds to

$$\frac{z_{\eta\eta}}{z} \ll k^2, \tag{5.34}$$

and Eq. (5.19) reduces to

$$(v_k)_{\eta\eta} - k^2 v_k = 0 \tag{5.35}$$

[17] In other words, one is applying again the Einstein equivalence principle (Will, 1993), this time to the physics of quantum fluctuations of the field ϕ.

in this limit, with solutions $v_k \propto e^{\pm ik\eta}$. Eq. (5.30) yields

$$\delta\phi_{\varphi k} = \frac{1}{a\sqrt{Z}\sqrt{2k}}\left[c_1(\vec{k})\, e^{ik|\eta|} + c_2(\vec{k})\, e^{-ik|\eta|}\right] \tag{5.36}$$

which can also be obtained by expansion of the solutions (5.32) for $k|\eta| \gg 1$. Obviously, the positive frequency solution at small scales is obtained by setting

$$c_1(\vec{k}) = 0, \qquad c_2(\vec{k}) = 1 \tag{5.37}$$

so that

$$\delta\phi(\eta, \vec{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3\vec{x}\, \left[c_2(\vec{k})e^{i(\vec{k}\cdot\vec{x}-k\eta)} + c_2^*(\vec{k})e^{i(-\vec{k}\cdot\vec{x}+k|\eta|)}\right]. \tag{5.38}$$

The *power spectrum* of a quantity $f(t, \vec{x})$ is defined as

$$\mathcal{P}(k, t) \equiv \frac{k^3}{2\pi^2} \int d^3r\, \langle f(\vec{x} + \vec{r}, t)\, f(\vec{x}, t)\rangle_{\vec{x}}\, e^{-i\vec{k}\cdot\vec{r}} = \frac{k^3}{2\pi^2} |\, f_k(t)|^2, \tag{5.39}$$

where $\langle\ \rangle_{\vec{x}}$ denotes an average over the spatial coordinates $\vec{x}$ and $f_k(t)$ are the coefficients of the Fourier expansion

$$f(t, \vec{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3\vec{k}\, [\, f_k(t)\, e^{i\vec{k}\cdot\vec{x}} + f_k^*(t)\, e^{-i\vec{k}\cdot\vec{x}}]. \tag{5.40}$$

The power spectrum of the gauge-invariant operator $\delta\hat{\phi}_\varphi$ is

$$\mathcal{P}_{\delta\hat{\phi}_\varphi}(k, t) = \frac{k^3}{2\pi^2} \int d^3r\, \langle 0|\delta\hat{\phi}_{\varphi k}(\vec{x} + \vec{r}, t)\delta\hat{\phi}_{\varphi k}(\vec{x}, t)|0\rangle_{\vec{x}}\, e^{-i\vec{k}\cdot\vec{r}}, \tag{5.41}$$

where $\langle 0|\hat{A}|0\rangle$ denotes the expectation value of the operator $\hat{A}$ on the vacuum state. One is interested in computing the power spectrum for large-scale perturbations, that is, perturbations that cross outside the horizon during inflation, subsequently remain "frozen" while outside the horizon, and only after the end of inflation, during the radiation- or the matter-dominated era reenter the horizon to seed the formation of structures. In the large scale limit the solution of Eq. (5.2) is

$$\delta\phi_\varphi(t, \vec{x}) = -\frac{\dot{\phi}}{H}\left[C(\vec{x}) - D(\vec{x}) \int^t dt'\, \frac{1}{a^3 Z}\, \frac{H^2}{\dot{\phi}}\right], \tag{5.42}$$

where $C(\vec{x})$ and $D(\vec{x})$ are, respectively, the coefficients of a growing component and of a decaying component that we neglect in the following. Accordingly, the solution $\delta\phi_{\varphi k}(\eta)$ in the large scale limit $k|\eta| \ll 1$ is

$$\delta\phi_{\varphi k}(\eta) = \frac{i\sqrt{|\eta|}\Gamma(\nu)}{2a\sqrt{\pi Z}} \left(\frac{k|\eta|}{2}\right)^{-\nu} [c_2(\eta) - c_1(\eta)] \tag{5.43}$$

for $\nu \neq 0$, where Γ denotes the gamma function. The power spectrum (5.41) therefore is given by

$$\mathcal{P}^{1/2}_{\delta\hat{\phi}_\varphi}(k,\eta) = \frac{\Gamma(\nu)}{\pi^{3/2} a|\eta|\sqrt{Z}} \left(\frac{k|\eta|}{2}\right)^{3/2-\nu} |c_2(\vec{k}) - c_1(\vec{k})| \tag{5.44}$$

for $\nu \neq 0$, while one obtains (Hwang, 1990b, 1996, 1997a; Hwang and Noh, 1996)

$$\mathcal{P}^{1/2}_{\delta\hat{\phi}_\varphi}(k,\eta) = \frac{2\sqrt{|\eta|}}{a\sqrt{Z}} \left(\frac{k}{2\pi}\right)^{3/2} \ln(k|\eta|)|c_2(\vec{k}) - c_1(\vec{k})| \tag{5.45}$$

for $\nu = 0$. Now, Eq. (5.42) yields (neglecting the decaying component),

$$C(\vec{x}) = -\frac{H}{\dot{\phi}}\delta\phi_\varphi(t,\vec{x}) \tag{5.46}$$

and therefore, using Eq. (5.39),

$$\mathcal{P}^{1/2}_C(k,t) = \left|\frac{H}{\dot{\phi}}\right| \mathcal{P}^{1/2}_{\delta\phi_\varphi}(k,t). \tag{5.47}$$

By combining Eqs. (4.56) and (5.46) one obtains

$$\varphi_{\delta\phi} = -\frac{H}{\dot{\phi}}\delta\phi_\varphi = C. \tag{5.48}$$

The relation between temperature fluctuations of the cosmic microwave background and the variable $C(\vec{x})$ is given in Hwang (1996) as

$$\frac{\delta T}{T} = \frac{C}{5} \tag{5.49}$$

and therefore the spectrum of temperature fluctuations is

$$\mathcal{P}^{1/2}_{\delta T/T}(k,t) = \frac{1}{5}\mathcal{P}^{1/2}_C(k,t) \tag{5.50}$$

and Eq. (5.47) yields

$$\mathcal{P}^{1/2}_{\delta T/T}(k,t) = \frac{1}{5}\left|\frac{H}{\dot{\phi}}\right| \mathcal{P}^{1/2}_{\delta\phi_\varphi}(k,t) \tag{5.51}$$

with $\mathcal{P}^{1/2}_{\delta\phi_\varphi}$ given by Eqs. (5.44) and (5.45).

The *spectral index* of scalar perturbations is defined as

$$n_s \equiv 1 + \frac{d\ln \mathcal{P}_{\delta\hat{\varphi}_{\delta\phi}}}{d\ln k}, \tag{5.52}$$

and Eq. (5.50) immediately yields

$$n_s = 1 + \frac{d\ln \mathcal{P}_C}{d\ln k}. \tag{5.53}$$

By using Eqs. (5.47) and (5.44) one obtains

$$n_s = 4 - 2\nu \qquad (\nu \neq 0); \tag{5.54}$$

in the following we do not need the expression for $\nu = 0$. To proceed we need extra input, which is connected to the validity of the assumption (5.24), which we now discuss. Hwang (1990b, 1996, 1997a) and Hwang and Noh (1996) proved that Eq. (5.24) is satisfied for pole-like inflation $a(t) \propto (t - t_0)^{-q} (q > 0)$, an expansion law appearing in pre-big bang cosmology related to low-energy string theory.[18] However, it turns out (a point not discussed in these works) that *Eq.* (5.24) *is satisfied in slow-roll inflation, to first order*. This is interesting for us because we know, from Section 4, that for suitable values of ξ there is a de Sitter attractor (4.14) for nonminimal coupling, and therefore that it makes sense to consider the slow-roll approximation.

Most models of ordinary inflation are set and solved in the context of the slow-roll approximation. In generalized inflation as well, the fact that slow-roll conditions satisfy Eq. (5.24) allows one to solve Eq. (5.19) for the perturbations.

Let us have a deeper look at Eq. (5.24) and at the value of n for the slow-roll approximation. The quantity $z_{\eta\eta}/z$ in Eq. (5.19) was computed exactly in Hwang (1990b, 1996, 1997a) and Hwang and Noh (1996) in terms of the slow-roll coefficients (4.53)–(4.56). Upon assuming that the ϵ_i be small and that their derivatives $\dot{\epsilon}_i$ can be neglected ($i = 1, \ldots, 4$), a situation mimicking the usual one (Lidsey *et al.*, 1997), one obtains to lowest order

$$\frac{z_{\eta\eta}}{z} = a^2 H^2 (2 - 2\epsilon_1 + 3\epsilon_2 - 3\epsilon_3 + 3\epsilon_4). \tag{5.55}$$

The standard relation

$$\eta \simeq -\frac{1}{aH} \frac{1}{1+\epsilon_1}, \tag{5.56}$$

yields Eq. (5.24) with

$$n = 2 + 3(-2\epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4) \tag{5.57}$$

and

$$\nu = \frac{3}{2} - 2\epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4. \tag{5.58}$$

Eqs. (5.54) and (5.58) yield the spectral index of scalar perturbations for generalized slow-roll inflation,

$$n_s = 1 + 2(2\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4), \tag{5.59}$$

[18] Hwang (1998a,b) is devoted, respectively, to the calculation of scalar and tensor perturbations in pre-big bang cosmology.

where the right hand side is computed at the time when the perturbations cross outside the horizon during inflation. The deviations of n_s from unity (i.e., from an exactly scale-invariant Harrison–Zeldovich spectrum) are small during generalized slow-rolling. For $\xi = 0$ one recovers the well-known formula for the spectral index of ordinary inflation (Lidsey *et al.*, 1997) $n_s = 1 - 4\epsilon_H + 2\eta_H$.

5.2. Gravitational Wave Perturbations

Tensor perturbations are generated as quantum fluctuations of the metric tensor g_{ab} in nonminimally coupled inflation, and were calculated in Hwang (1998a) with a procedure similar to the one used for scalar perturbations. We briefly review also this calculation.

It is convenient to introduce tensor perturbations as the trace-free and transverse quantities c_{ij} in the metric

$$ds^2 = -dt^2 + a^2(t)(\delta_{ij} + 2c_{ij})\, dx^i\, dx^j, \tag{5.60}$$

with

$$c_i^i = 0, \quad c_{ij}{}^{,j} = 0. \tag{5.61}$$

The power spectrum is

$$\mathcal{P}_{c_{ij}}(k,t) = \frac{k^3}{2\pi^2}\int d^3\vec{r}\ \langle c_{ij}(\vec{x}+\vec{r},t)c_{ij}(\vec{x},t)\rangle_{\vec{x}}\ e^{-i\vec{k}\cdot\vec{r}} \tag{5.62}$$

and the spectral index of tensor perturbations is

$$n_T = \frac{d\ln\mathcal{P}_{c_{ij}}}{d\ln k}. \tag{5.63}$$

One obtains

$$\mathcal{P}^{1/2}_{c_{ij}}(k,\eta) = \frac{8\pi GH}{\sqrt{2}\pi}\frac{1}{\sqrt{1-\xi k^2\phi^2}}\frac{\Gamma(\nu)}{\Gamma(3/2)}\left(\frac{k|\eta|}{2}\right)^{3/2-\nu_g} \times \sqrt{\frac{1}{2}\sum_l |c_{l1}(\vec{k}) - c_{l2}(\vec{k})|^2}, \tag{5.64}$$

where the summation $\sum_l$ is intended over the two polarization states $\times$ and $+$ of gravitational waves. In the slow-roll approximation one has

$$\frac{z_{\eta\eta}}{z} = \frac{m}{\eta^2}, \tag{5.65}$$

where m is the linear combination of the slow-roll parameters

$$m = 2 - 3(\epsilon_1 - \epsilon_3) \tag{5.66}$$

and $\nu_g = (m + 1/4)^{1/2}$, as usual. Hence, $\nu_g \simeq 3/2 - \epsilon_1 + \epsilon_3$ and, to first order,

$$n_T = 2(\epsilon_1 - \epsilon_3). \tag{5.67}$$

Eq. (5.67) reduces to the well known spectral index of tensor perturbations (Lidsey *et al.*, 1997) of ordinary inflation when $\xi \to 0$. Note that n_T is very small for slow-roll inflation, like in slow-roll ordinary inflation.

This completes the review of the calculation of spectral indices in Hwang (1990b, 1996, 1997a) and Hwang and Noh (1996); the reader is invited to consult the relevant papers.[19] We just reported on the published work and completed the calculation corresponding to the slow-roll regime. Slow-roll inflation is not explicitly mentioned in Hwang (1990b, 1996, 1997a) and Hwang and Noh (1996), probably because the attractor role of de Sitter solutions was not established at that time. The knowledge that slow-roll inflation does indeed make sense for $\xi \neq 0$ allows us to claim that Hwang's (Hwang, 1990b, 1996, 1997a; Hwang and Noh, 1996) calculation applies to it, and to use Eqs. (5.59) and (5.67) for the spectral indices to test generalized inflation with observations of the cosmic microwave background.

6. OPEN PROBLEMS

The program of rethinking inflationary theory by including the (generally unavoidable) NMC of the scalar field, a crucial ingredient too often forgotten, is not exhausted by the results presented in the previous sections. In this section, we outline open problems that constitute avenues for future research.

It may be useful to remark that, in addition to inflation, NMC changes the description and the results of the dynamical systems approach to cosmology (e.g., Gunzig *et al.*, 2000, in press), quantum cosmology (Barvinsky, 1999; Barvinsky *et al.*, 1977; da Silva and Williams, 2000; Fakir, 1990; Kamenshchik *et al.*, 1995; Okamura, 1998), classical and quantum wormholes (Coule, 1992; Coule and Maeda, 1990; Halliwell and Laflamme, 1989), and constitutes a line of approach to the cosmological constant problem (Dolgov, 1983; Ford, 1987; Suen and Will, 1988).

6.1. Doppler Peaks

We did not present Doppler peaks for generalized inflation: the acoustic oscillations well known for minimal coupling (Liddle and Lyth, 1993) are indeed modified by NMC. Although a plot of the Doppler peaks requires the specification

[19] A synthesis is given in Kaiser (1995b) using different variables.

of a particular scenario of generalized inflation (the potential $V(\phi)$, the value of the coupling constant ξ, details about the end of inflation and reheating, etc.), preliminary work was done in Baccigalupi *et al.* (2000) and Perrotta *et al.* (1999). The acoustic peaks and the spectrum turnover are displaced, and the effects of NMC in this model are (Baccigalupi *et al.*, 2000; Perrotta *et al.*, 1999)

(i) an enhancement of the large scale, low multipoles l region of the Doppler peaks, due to an enhancement of the integrated Sachs–Wolfe (or Rees–Sciama) effect;
(ii) the oscillating region of the Doppler peaks is attenuated;
(iii) the location of the peaks is shifted toward higher multipoles.

These features were derived by direct integration of the equations for the perturbations with a modified CMBFAST code, under the assumption that the scalar field is the source of quintessence and that it has a potential of the form $V(\phi) \propto 1/\phi^{\alpha} (\alpha > 0)$ in the range of values spanned by ϕ today. Similar qualitative effects appear in a model based on induced gravity, and are interpreted as the signature of a broad class of scalar–tensor gravity theories in the cosmic microwave background (Baccigalupi *et al.*, 2000; Perrotta *et al.*, 1999).

However, the nonminimally coupled scalar field driving inflation does not necessarily have to be identified with the same scalar which possibly constitutes quintessence today (as is instead done in quintessential inflation). Such an identification, indeed, may appear artificial. If the quintessence and the inflation field do not coincide, the rather strong constraint (Baccigalupi *et al.*, 2000; Chiba, 1999; Perrotta *et al.*, 1999) $\xi^2 G\phi^2_{\text{today}} < 1/500$ coming from tests of gravity in the Solar System (Will, 1993) is circumvented. More important, the features of the Doppler peaks could then be different. A separate study is needed to explore their features in detail.

6.2. Cosmic No-Hair Theorems

Cosmic no-hair theorems (Kolb and Turner, 1994) in the presence of NMC are not known: one would like to know whether inflation is still a generic phenomenon when $\xi \neq 0$. In other words, starting with an anisotropic Bianchi model, does inflation occur and lead the universe toward a highly spatially homogeneous and isotropic FLRW state, with flat spatial sections?

Preliminary results (Futamase *et al.*, 1989; Starobinsky, 1980a) show that the convergence to the $k = 0$ FLRW universe can disappear by going from $\xi = 0$ to $\xi \neq 0$. Our perturbation analysis of Section 4 shows that, when the deviations from homogeneity and isotropy are small, the de Sitter solutions are still inflationary attractors in the phase space, but an analysis for large deviations from a FLRW space is needed.

6.3. Reconstruction of the Inflationary Potential

The reconstruction of the inflationary potential from cosmological observations is a task that was undertaken only for $\xi = 0$. Given the unavoidability of NMC in the general case, one would like to have a similar formalism also for $\xi \neq 0$. Nothing has been done yet on the subject.

6.4. A Connection Between Inflation and Boson Stars?

Finally, we would like to point out a possible connection between generalized inflation and relativistic astrophysics. It appears from observations of gravitational microlensing that there is a population of objects with mass $M < M_\odot$ responsible for the observed microlensing events. Boson stars (see Jetzer, 1992 for a review) are candidates; nowadays, they are not regarded as unrealistic objects.

If boson stars exist at all, they are relics from the early universe since they are formed by bosons that were around at primordial times and aggregated to form balls very early. Hence, there is the possibility that boson stars are formed of a condensate of inflation particles. Since the stability of boson stars depends on the value of ξ (van der Bij and Gleiser, 1987), and such objects can only exists in a certain range of values of ξ (Jetzer, 1992; van der Bij and Gleiser, 1987), one can connect this range of stability to successful generalized inflation scenarios (of which only a few have been studied in the literature). This connection could be a link between present-day objects and the early universe; work is in progress in this direction. For example, gravitational lensing by boson stars was studied for $\xi = 0$ and shown to have characteristic signatures (Dabrowski and Schunck, in press); this study could be generalized to the $\xi \neq 0$ case.

To conclude, the inclusion of NMC in the equations of inflation seems to be necessary in most inflationary theories, and leads to important consequences. Inflationary solutions are changed into noninflationary ones, and fine-tuning problems appear. The much needed slow-roll approximation to inflation is meaningful only when particular relations between the scalar field potential, its derivatives, and the value of ξ are satisfied. In generalized slow-roll inflation, the spectra of density and gravitational wave perturbations have been computed, and are given in Section 5. In our opinion, the most interesting problems left open in generalized inflation are whether cosmic no-hair theorems hold, and the reconstruction of the inflationary potential, which will be the subjects of future research.

APPENDIX A

The Bessel function $J_\nu(s)$ can be expressed as

$$J_\nu(s) = \frac{H_\nu^{(1)}(s) + H_\nu^{(2)}(s)}{2}, \tag{A.1}$$

and therefore

$$v_k(\eta) = \sqrt{k\eta}\, J_\nu(k\eta) = \frac{\sqrt{k\eta}}{2}\left[H_\nu^{(1)}(k\eta) + H_\nu^{(2)}(k\eta)\right]. \tag{A.2}$$

In addition, the property

$$J_p(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(p+k+1)} \left(\frac{z}{2}\right)^{p+2k} \tag{A.3}$$

yields

$$J_p(-z) = (-1)^p J_p(z). \tag{A.4}$$

Then, $J_\nu(k\eta) = (-1)^\nu J_\nu(k|\eta|)$ if $\eta < 0$.

APPENDIX B

In de Sitter space, the dependence of the scale factor on the comoving time t

$$a = a_0 \exp(H_0 t) \tag{B.1}$$

and the definition of conformal time

$$\eta = \int^t \frac{dt'}{a(t')}, \tag{B.2}$$

yield the relation

$$\eta = -\frac{1}{aH_0} = -\frac{e^{-H_0 t}}{a_0 H_0}. \tag{B.3}$$

For *expanding* de Sitter spaces ($H_0 > 0$), $t \to +\infty$ corresponds to $\eta \to 0$, while for *contracting* ($H_0 < 0$) de Sitter spaces, $t \to +\infty$ corresponds to $\eta \to +\infty$. During slow-roll inflation, Eq. (B.3) is corrected to

$$\eta = -\frac{1}{aH}\frac{1}{1+\epsilon_1}, \tag{B.4}$$

which holds in the approximation that the derivatives of the slow-roll parameters ϵ_i can be neglected (Hwang, 1997b, 1998a).

ACKNOWLEDGMENTS

It is a pleasure to thank S. Sonego, M. Bruni, E. Gunzig, S. Odintsov, C. Baccigalupi, V. Sahni, L. Ford, J. Miller, S. Liberati, and E. Calzetta for useful discussions. This work was supported by the grants EC-DGXII PSS* 0992 and HPHA-CT-2000-00015 and by OLAM, Fondation pour la Recherche Fondamentale, Brussels.

REFERENCES

Abbott, L. F. (1981). *Nuclear Physics B* **185**, 233.

Allen, B. (1993). *Nuclear Physics B* **226**, 232.

Amendola, L., Litterio, M., and Occhionero, F. (1998). *International Journal of Modern Physics A* **5**, 3861.

Baccigalupi, C., Matarrese, S., and Perrotta, F. (2000). *Phisical Review D* **62**, 123510.

Bardeen, J. M. (1980). *Physical Review D* **22**, 1882.

Barroso, A., Casasayas, J., Crawford, P., Woniz, P., and Nunes, A. (1992). *Physics Letters B* **275**, 264.

Barvinsky, A. O. (1999). *Nuclear Physics B* **561**, 159.

Barvinsky, A. O., Kamenshchik, A. Yu., and Mishakov, I. V. (1977). *Nuclear Physics B* **491**, 378.

Bassett, B. and Liberati, S. (1998). *Physical Review D* **58,** 021302.

Birrell, N. D. and Davies, P. C. (1980a). *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge.

Birrell, N. D. and Davies, P. C. (1980b). *Physical Review D* **22**, 322.

Bonanno, A. (1995). *Physical Review D* **52**, 969.

Bonanno, A. and Zappalá, D. (1997). *Physical Review D* **55**, 6135.

Buchbinder, I. L. (1986). *Fortschritte der Physik* **34**, 605.

Buchbinder, I. L. and Odintsov, S. D. (1983). *Soviet Journal of Nuclear Physics* **40**, 848.

Buchbinder, I. L. and Odintsov, S. D. (1985). *Lettere al Nuovo Cimento* **42**, 379.

Buchbinder, I. L., Odintsov, S. D., and Shapiro, I. L. (1986). *Group-Theoretical Methods in Physics*, M. Markov, ed., Nauka, Moscow.

Buchbinder, I., Odintsov, S. O., and Lichzier, I. (1989). *Classical and Quantum Gravity* **6**, 605.

Buchbinder, I., Odintsov, S. O., and Shapiro, I. (1992). *Effective Action in Quantum Gravity*, IOP Publishing, Bristol.

Callan, C. G., Jr., Coleman, S., and Jackiw, R. (1970). *Annals of Physics* (*New York*) **59**, 42.

Chernikov, N. A. and Tagirov, E. A. (1968). *Annales de Institut Henri Poincaré A* **9**, 109.

Chiba, T. (1999). *Physical Review D* **60**, 083508.

Coule, D. H. (1992). *Classical and Quantum Gravity* **9**, 2353.

Coule, D. H. and Maeda, K. (1990). *Classical and Quantum Gravity* **7**, 955.

Dabrowski, M. P. and Schunck, F. E. (in press). *Astrophysical Journal* **535**, 316.

da Silva, C. and Williams, R. M. (2000). *Classical and Quantum Gravity* **17**, 1827.

DeWitt, B. S. and Brehme, R. W. (1960). *Annals of Physics* (*New York*) **9**, 220.

Dolgov, A. D. (1983). *The Very Early Universe*, G. W. Gibbons, S. W. Hawking, and T. C. Siklos, eds., Cambridge University Press, Cambridge.

Easther, R. (1996). *Classical and Quantum Gravity* **13**, 1775.

Elizalde, E. and Odintsov, S. D. (1994). *Physics Letters B* **333**, 331.

Ellis, G. F. R. and Bruni, M. (1989). *Physical Review D* **40**, 1804.

Ellis, G. F. R., Bruni, M., and Hwang, J. (1990). *Physical Review D* **42**, 1035.

Ellis, G. F. R., Hwang, J., and Bruni, M. (1989). *Physical Review D* **40**, 1819.

Fakir, R. (1990). *Physical Review D* **41**, 3012.

Fakir, R. and Habib, S. (1993). *Modern Physics Letters A* **8**, 2827.

Fakir, R., Habib S., and Unruh, W. G. (1992). *Astrophysical Journal* **394**, 396.

Fakir, R. and Unruh, W. G. (1990a). *Physical Review D* **41**, 1783.

Fakir, R. and Unruh, W. G. (1990b). *Physical Review D* **41**, 1792, 3012.

Faraoni, V. (1996). *Physical Review D* **53**, 6813.

Faraoni, V. (1998). *Proceedings of the Seventh Canadian Conference on General Relativity and Relativistic Astrophysics*, Calgary, Canada, 1997, D. Hobill, ed., University of Calgary Press, Calgary.

Faraoni, V. (1999). *American Journal of Physics* **67**, 732.

Faraoni, V. (2000a). *Physical Review D* **62**, 023504.
Faraoni, V. (2000b). *Physics Letters A* **269**, 209.
Faraoni, V. and Cooperstock, F. I. (1998). *European Journal of Physics* **19**, 419.
Faraoni, V. and Gunzig, E. (1998). *International Journal of Modern Physics D* **8**, 177.
Faraoni, V. and Sonego, S. (1994). *Proceedings of the Fifth Canadian Conference on General Relativity and Relativistic Astrophysics*, Waterloo, Canada, 1993, R. B. Mann and R. G. McLenaghan, eds., World Scientific, Singapore, p. 386.
Ford, L. H. (1987). *Physical Review D* **32**, 1409.
Ford, L. H. and Toms, D. J. (1982). *Physical Review D* **25**, 1510.
Friedlander, F. G. (1975). *The Wave Equation on a Curved Spacetime*, Cambridge University Press, Cambridge.
Fukuyama, T., Hatakeyama, M., Miyoshi, M., Morikawa, M., and Nakamichi, A. (1997). *International Journal of Modern Physics D* **6**, 69.
Futamase, T. and Maeda, K. (1989). *Physical Review D* **39**, 399.
Futamase, T., Rothman, T., and Matzner, R. (1989). *Physical Review D* **39**, 405
Futamase, T. and Tanaka, M. (1999). *Physical Review D* **60**, 063511.
Garcia-Bellido, J. and Linde, A. (1995). *Physical Review D* **52**, 6370.
Grib, A. A. and Poberii, E. A. (1995). *Helvetica Physica Acta* **68**, 380.
Grib, A. A. and Rodrigues, W. A. (1995). *Gravitation and Cosmology* **1**, 273.
Gunzig, E., Faraoni, V., Rocha Filho, T. M., Figueiredo, A., and Brenig, L. (2000). *Classical and Quantum Gravity* **17**, 1783.
Gunzig, E., Faraoni, V., Rocha Filho, T. M., Figueiredo, A., and Brenig, L. (2000). Peyresq Physics Proceedings IV, Peyresq, France, 1999. *International Journal of Theoretical Physics* **39**, 1901. (Special issue).
Guth, A. H. (1997). *The Inflationary Universe*, Addison-Wesley, Reading, MA.
Hadamard, J. (1952). *Lectures on Cauchy's Problem in Linear Partial Differential Equations*, Dover, New York.
Halliwell, J. (1987). *Physics Letters B* **185**, 341.
Halliwell, J. J. and Laflamme, R. (1989). *Classical and Quantum Gravity* **6**, 1839.
Hill, C. T. and Salopek, D. S. (1992). *Annals of Physics* (*New York*) **213**, 21.
Hiscock, W. A. (1990). *Classical and Quantum Gravity* **7**, L35.
Hosotani, Y. (1985). *Physical Review D* **32**, 1949.
Hwang, J. (1990a). *Classical and Quantum Gravity* **7**, 1613.
Hwang, J. (1990b). *Physical Review D* **42**, 2601.
Hwang, J. (1996). *Physical Review D* **53**, 762.
Hwang, J. (1997a). *Classical and Quantum Gravity* **14**, 1981.
Hwang, J. (1997b). *Classical and Quantum Gravity* **14**, 3327.
Hwang, J. (1998a). *Classical and Quantum Gravity* **15**, 1401.
Hwang, J. (1998b). *Classical and Quantum Gravity* **15**, 1387.
Hwang, J. and Noh, H. (1996). *Physical Review D* **54**, 1460.
Hwang, J.-C. and Noh, H. (1998). *Physical Review Letters* **81**, 5274.
Hwang, J. and Vishniac, E. T. (1990). *Astrophysical Journal* **353**, 1.
Ishikawa, J. (1983). *Physical Review D* **28**, 2445.
Jetzer, P. (1992). *Physics Reports* **220**, 1636.
Kaiser, D. I. (1995a). *Physical Review D* **52**, 4295.
Kaiser, D. I. (1995b). Preprint astro-ph/9507048. (unpublished).
Kamenshchik, A. Y., Khalatnikov, I. M., and Toporensky, A. V. (1995). *Physics Letters B* **357**, 35.
Kolb, E. W. and Turner, M. S. (1994). *The Early Universe*, Addison-Wesley, Reading MA.
Komatsu, E. and Futamase, T. (1998). *Physical Review D* **58**, 023004.
Komatsu, E. and Futamase, T. (1999). *Physical Review D* **59**, 064029.

Laycock, A. M. and Liddle, A. R. (1994). *Physical Review D* **49**, 1827.
Lee, J. *et al.* (1999). *Physical Review D* **61**, 027301.
Liddle, A. R. (1999). Preprint astro-ph/9901124. In *Proceedings of the 1999 Summer School in High Energy Physics and Cosmology*, A. Masiero, G. Senjanovic, and A. Smirnov, eds., (World Scientific, Singapore, 1999).
Liddle, A. R. and Lyth, D. H. (1993). *Physics Reports* **231**, 1.
Liddle, A. A., Parsons, P., and Barrow, J. D. (1994). *Physical Review D* **50**, 7222.
Lidsey, J. E., Liddle, A. R., Kolb, E. W., Copeland, E. J., Barreire, T., and Abney, M. (1997). *Reviews of Modern Physics* **69**, 373.
Linde, A. D. (1990). *Particle Physics and Inflationary Cosmology*, Hardwood Academic, Chur, Switzerland.
Madsen, M. (1988). *Classical and Quantum Gravity* **5**, 627.
Maeda, K. (1989). *Physical Review D* **39**, 2848.
Makino, N. and Sasaki, M. (1991). *Progress of Theoretical Physics* **86**, 103
Martin, J. and Schwarz, D. J. (2001). *Physics Letters B* **500**, 1.
Morikawa, M. (1990). *Astrophysical Journal Letters* **362**, L37.
Morikawa, M. (1991). *Astrophysical Journal* **369**, 20.
Mukhanov, V. F., Feldman, H. A., and Brandenberger, R. H. (1992). *Physics Reports* **215**, 203.
Muller, V., Schmidt, H.-J., and Starobinsky, A. A. (1990). *Classical and Quantum Gravity* **7**, 1163.
Muta, T. S. and Odintsov, S. D. (1991). *Modern Physics Letters A* **6**, 3641.
Nelson, B. and Panangaden, P. (1982). *Physical Review D* **25**, 1019.
Odintsov, S. D. (1991). *Fortschritte der Physik* **39**, 621.
Okamura, T. (1998). *Proceedings of the Asia-Pacific Conference on Gravitation and Cosmology*, Seoul, Korea, 1996, Y. M. Cho, C. H. Lee, and S.-W. Kim, eds., World Scientific, Singapore.
Parker, L. and Toms, D. J. (1985a). *Physical Review D* **29**, 1584.
Parker, L. and Toms, D. J. (1985b). *Physical Review D* **32**, 1409.
Perlmutter, S. *et al.* (1998). *Nature* **391**, 51.
Perrotta, F., Baccigalupi, C., and Matarrese, S. (1999). *Physical Review D* **61**, 023507.
Reuter, M. (1994). *Physical Review D* **49**, 6379.
Riess, A. G. *et al.* (1998). *Astrophysical Journal* **116**, 1009.
Rocha Filho, T. M., Figueiredo, A., Brenig, L., Gunzig, E., and Faraoni, V. (2000). Peyresq Physics Proceedings IV, Peyresq, France, 1999, *International Journal of Theoretical Physics* **39**, 1933. (Special issue).
Salopek, D. S., and Bond, J. R. (1990). *Physical Review D* **42**, 3936.
Salopek, D. S., Bond, J. R., and Bardeen, J. M. (1989). *Physical Review D* **40**, 1753.
Smoot, G. F. *et al.* (1992). *Astrophysical Journal Letters* **396**, L1.
Sonego, S. and Faraoni, V. (1993). *Classical and Quantum Gravity* **10**, 1185.
Starobinsky, A. A. (1980a). *Physics Letters B* **91**, 99.
Starobinsky, A. A. (1980b). *Soviet Astronomy Letters* **10**, 135.
Starobinsky, A. A. (1981). *Soviet Astronomy Letters* **7**, 36.
Steinhardt, P. J., Wang, L., and Zlatev, I. (1999). *Physical Review D* **59**, 123504.
Suen, W.-M. and Will, C. M. (1988). *Physics Letters B* **205**, 447.
Uzan, J.-P. (1999). *Physical Review D* **59**, 123510.
van der Bij, J. J. and Gleiser, M. (1987). *Physics Letters B* **194**, 482.
van der Bij, J. J. and Radu, E. (2000). Preprint hep-th/0003073.
Voloshin, M. B. and Dolgov, A. D. (1982). *Soviet Journal of Nuclear Physics* **35**, 120.
Wald, R. M. (1984). *General Relativity*, University of Chicago Press, Chicago.
Will, C. M. (1993). *Theory and Experiment in Gravitational Physics*, revised edn. Cambridge University Press, Cambridge.
Zlatev, I., Wang, L., and Steinhardt, P. J. (1999). *Physical Review Letters* **82**, 896.